CAPTIONS FOR FIGURES

1a) IPs P_s^- and P_s^+
1b) IF F_s^-
1c) IF F_0

2a) X
2b) $\bar{X}$

3a) $x \in L(\sigma)$
3b) $x \in L(\sigma)$
3c) $x \notin L(\sigma)$

4a) M
4b) $\hat{M}$
4c) neighborhood of P_0^+

5) why $p \ll x$

6) if $x \notin L(\sigma)$

7) action of f

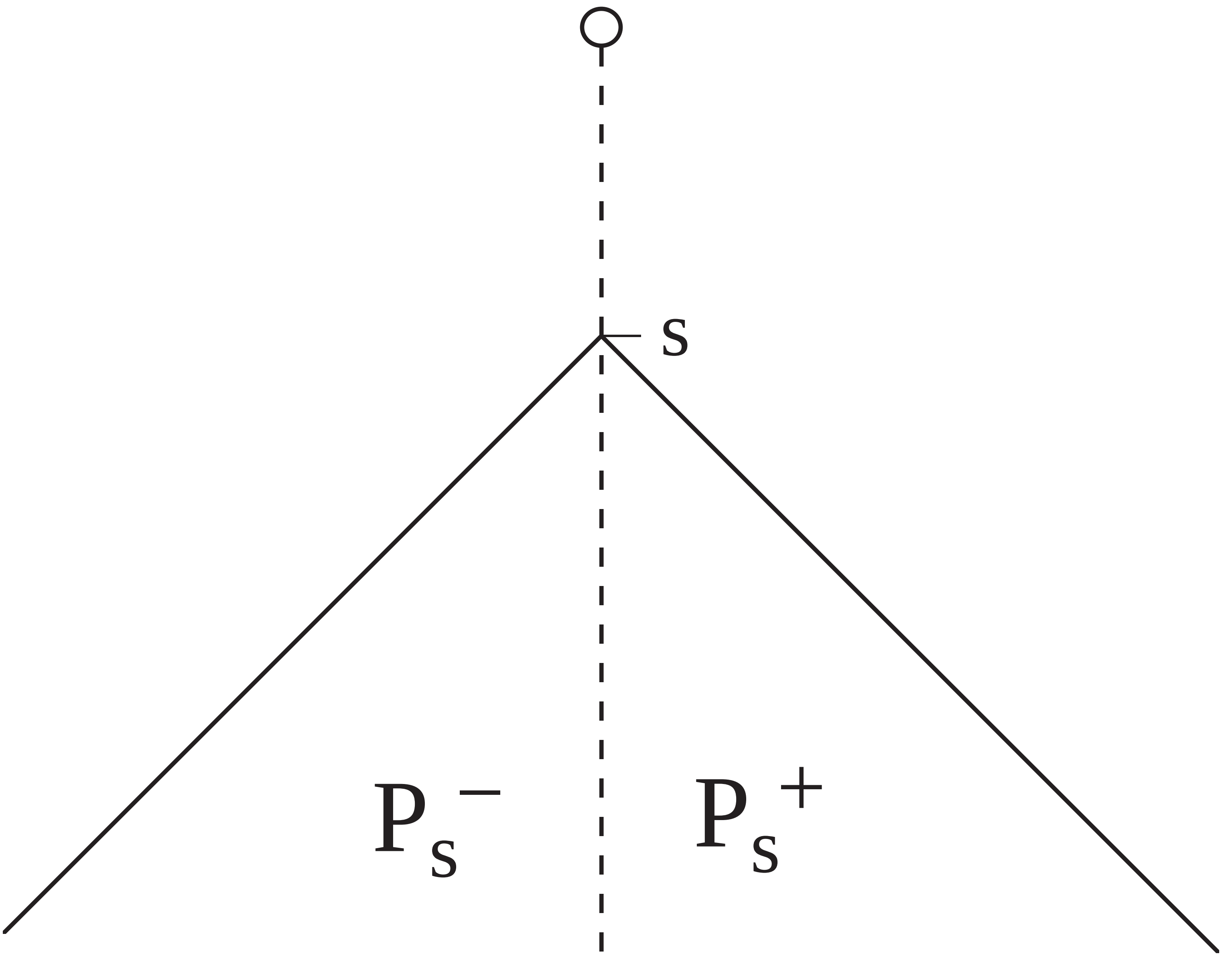
8) τ^0 should converge to B_0

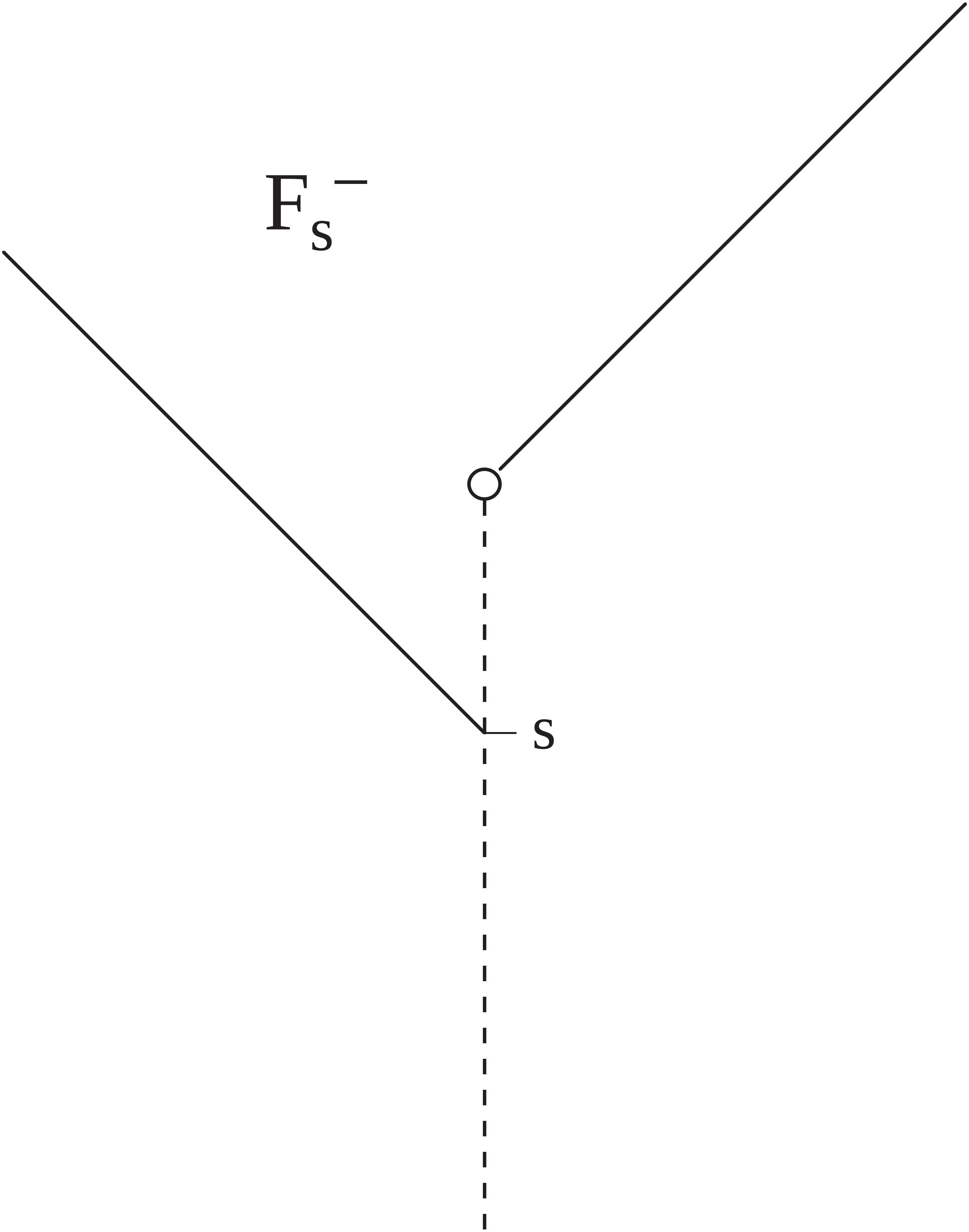
9a) X
9b) $\hat{X}$
9c) Y
9d) $\hat{Y}$

10) IPs in Y

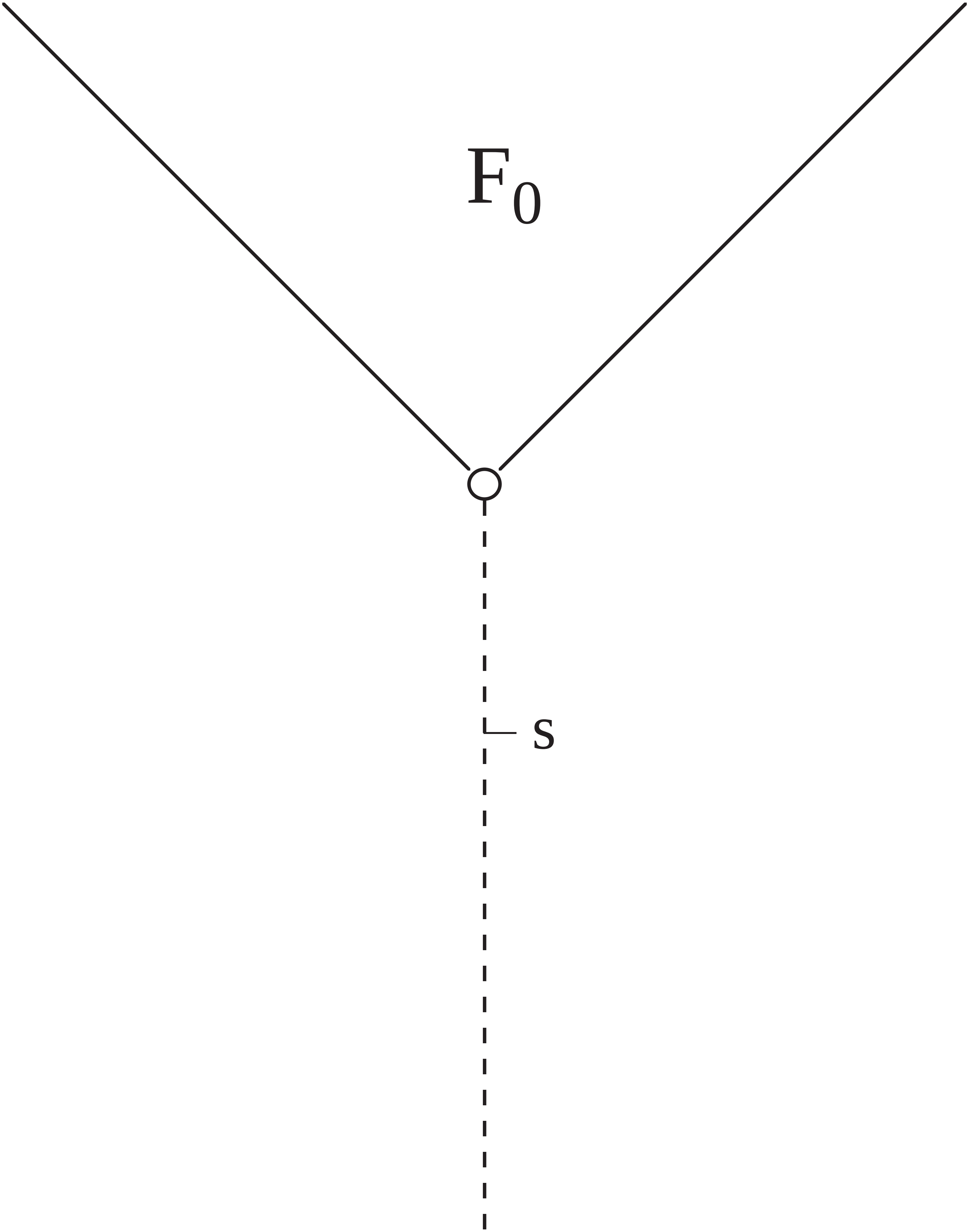
11) why $P_n \subset Q_{n+1}$

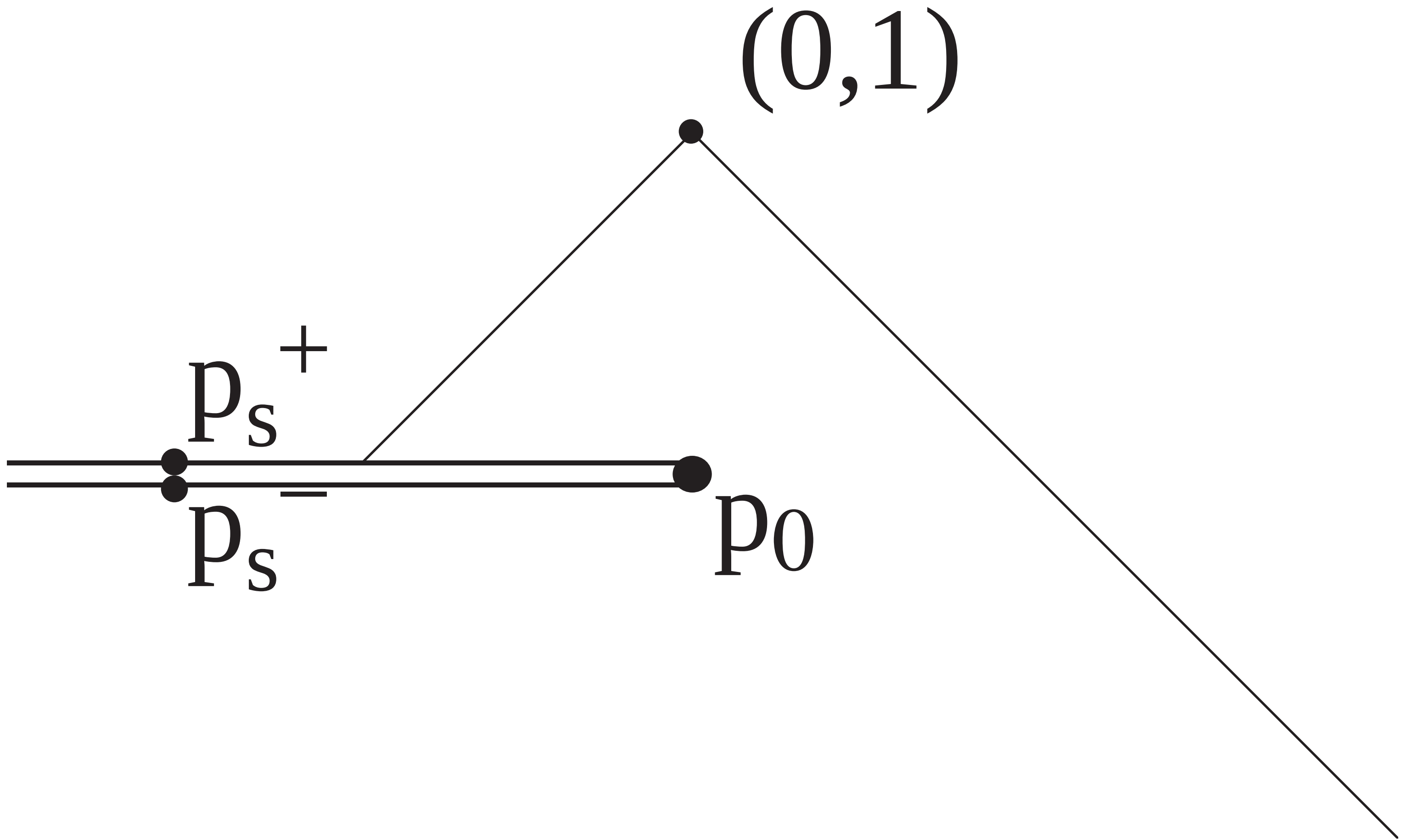
12) why $Q \subset \bar{I}^-(p)$

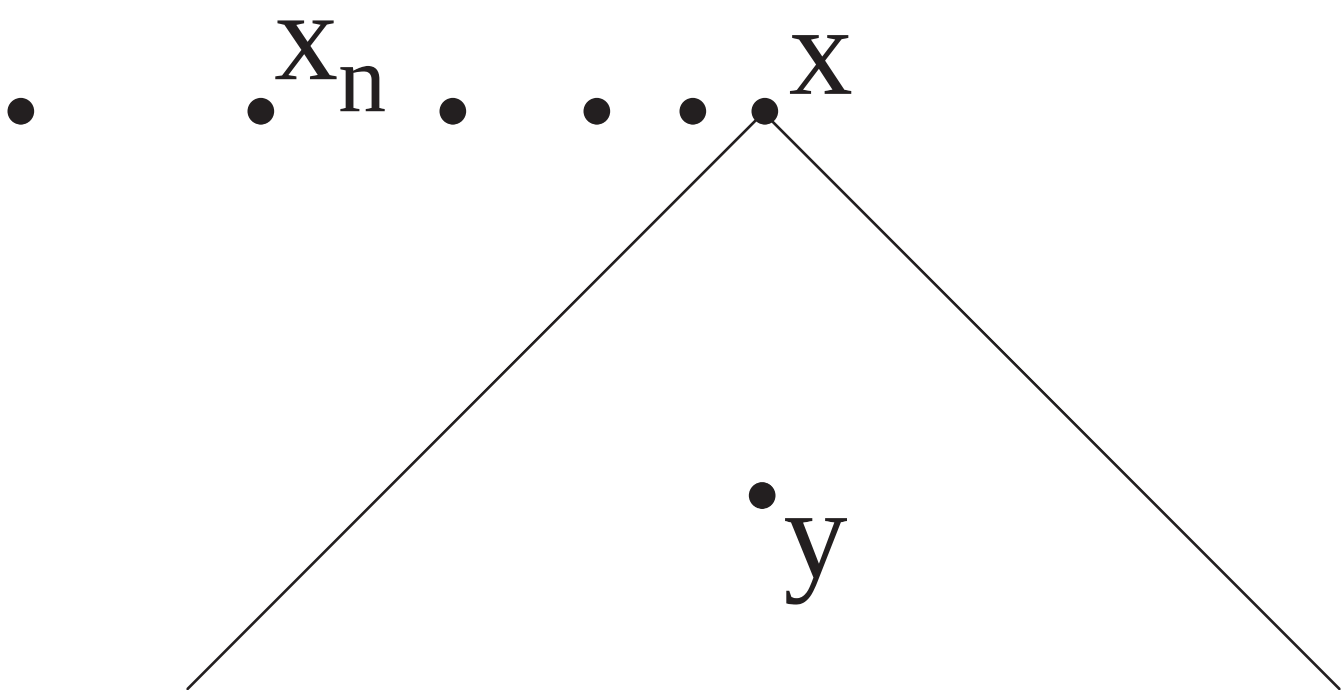

s
P_s^-
P_s^+



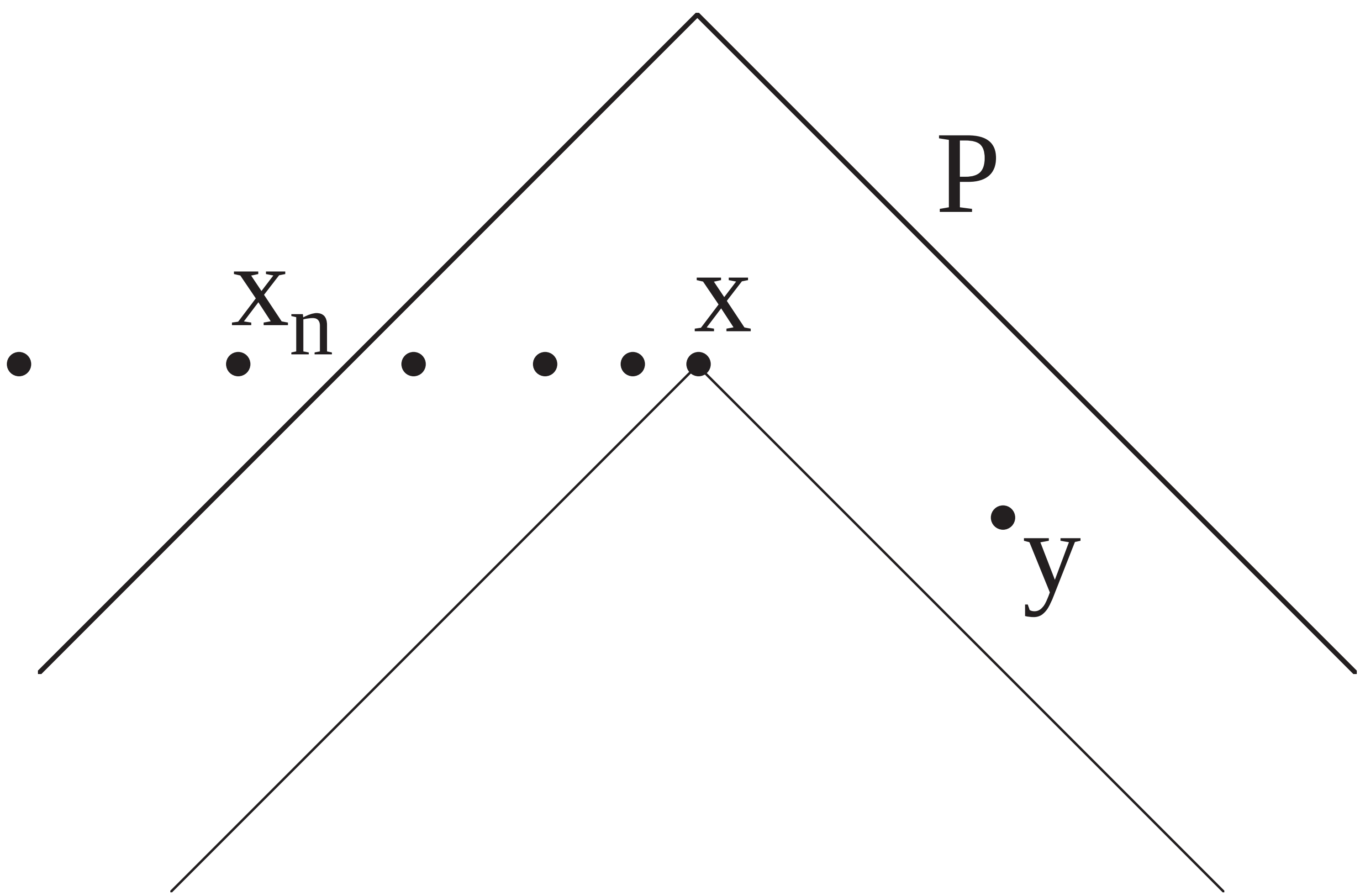
F_s^-
s


F_0
s

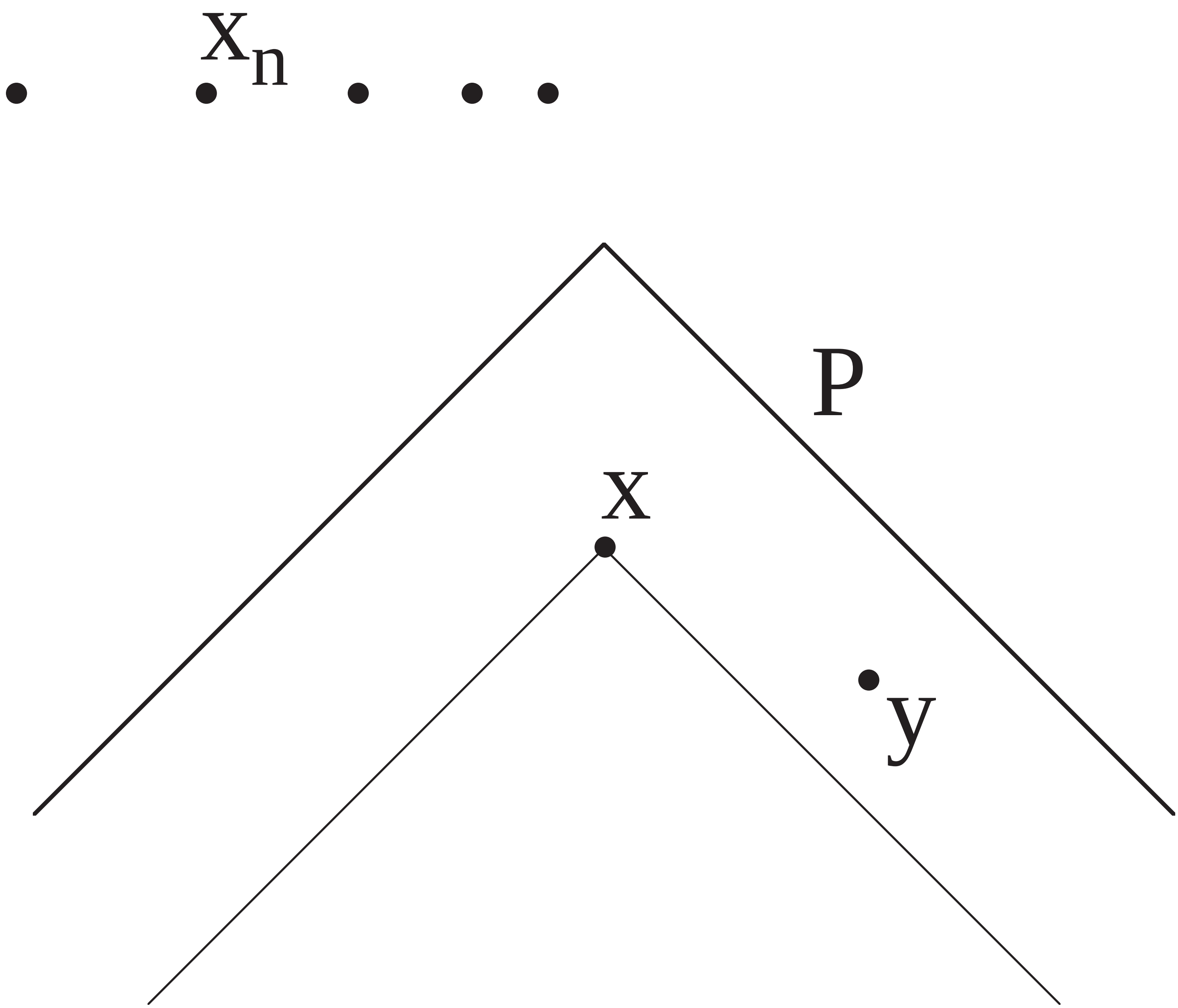

(0,1)
p_s^+
p_s^-
p_0



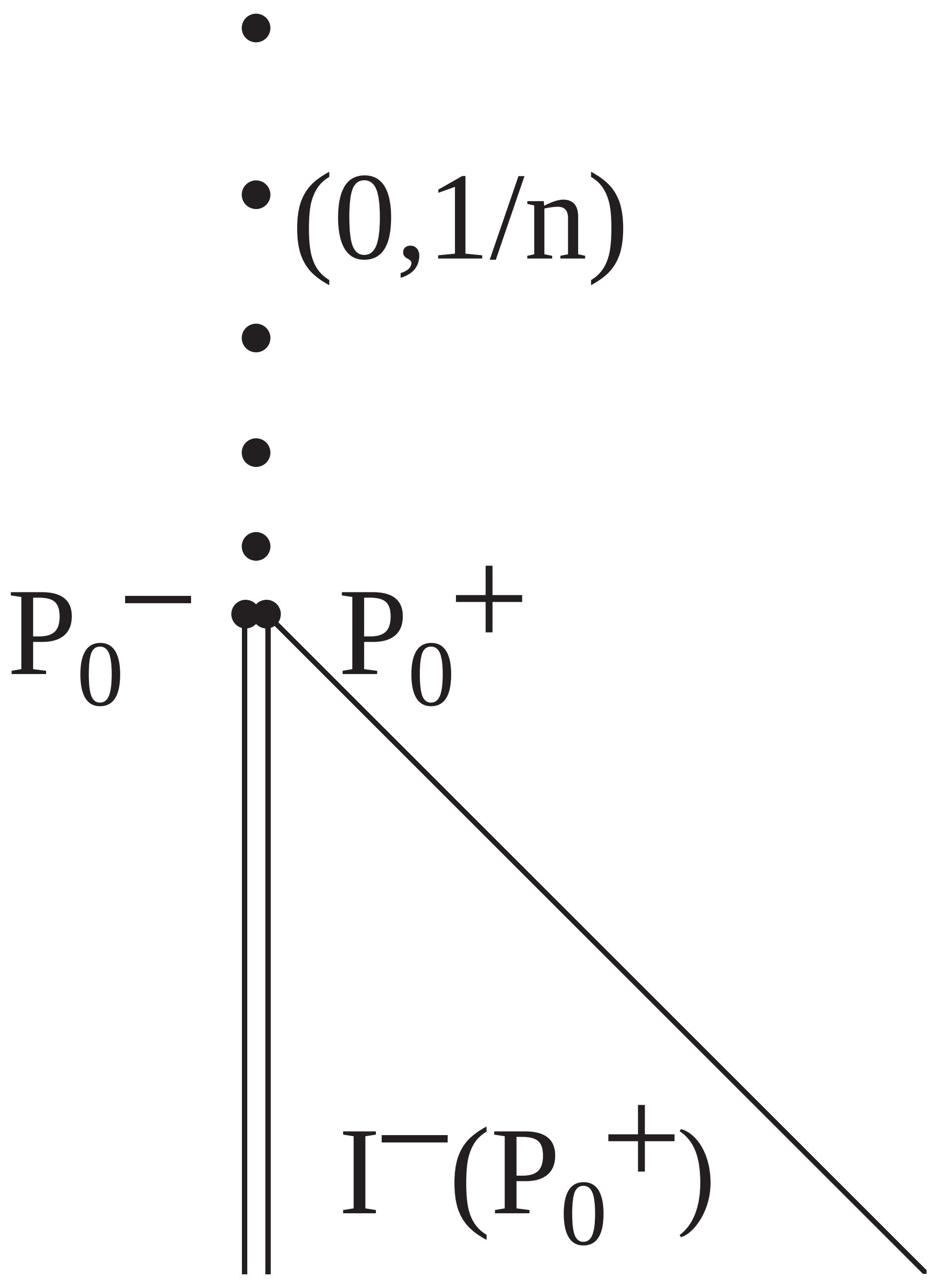
x_n
x
y



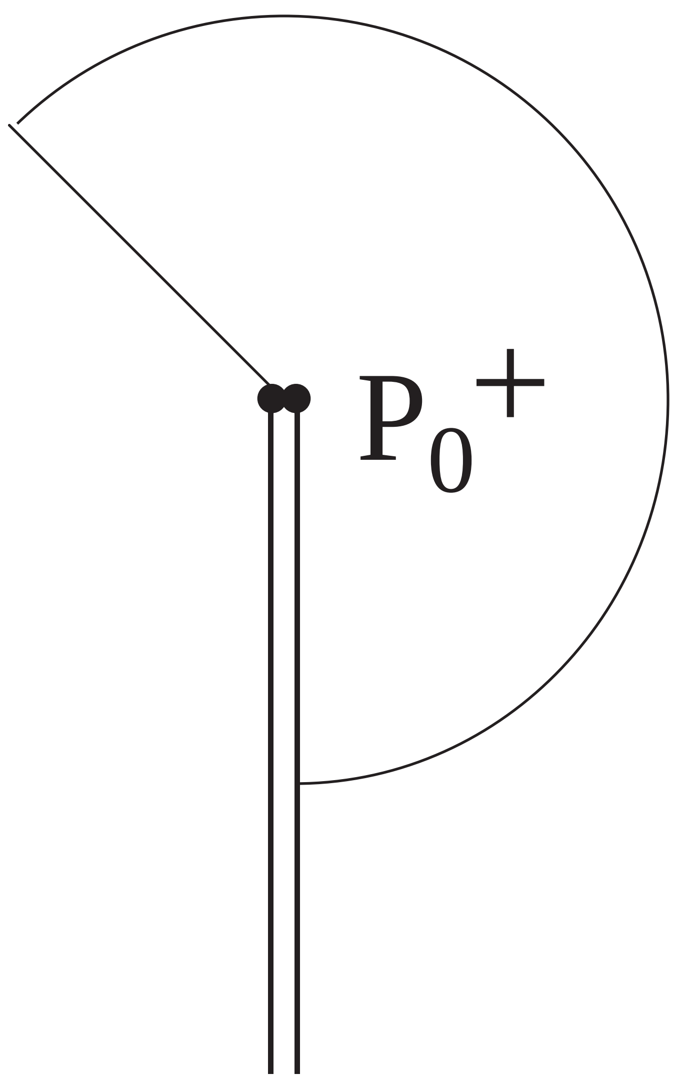
P
x_n
x
y

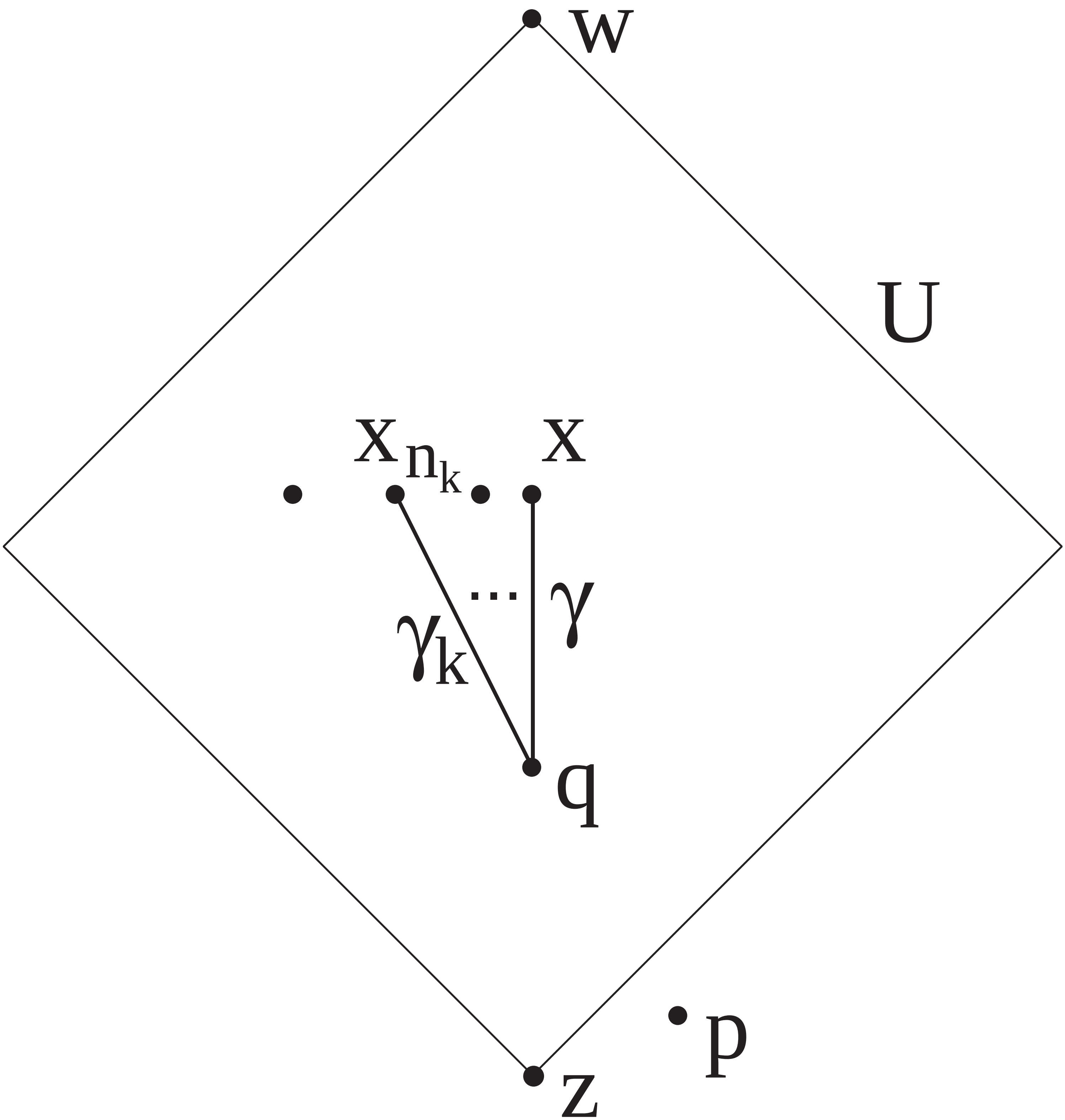


x_n
P
x
y

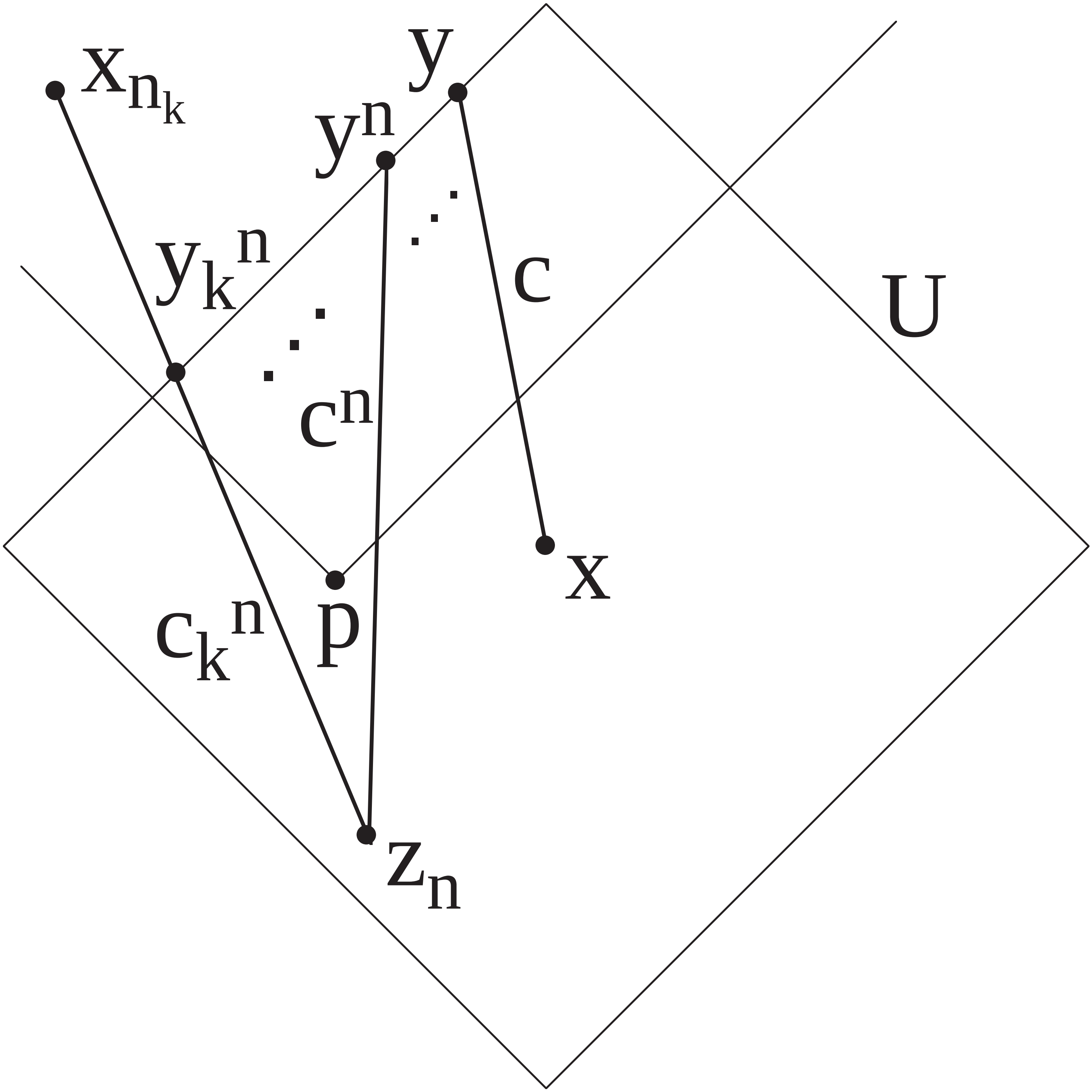


(0,1/n)
P0−
P0+
I−(P0+)

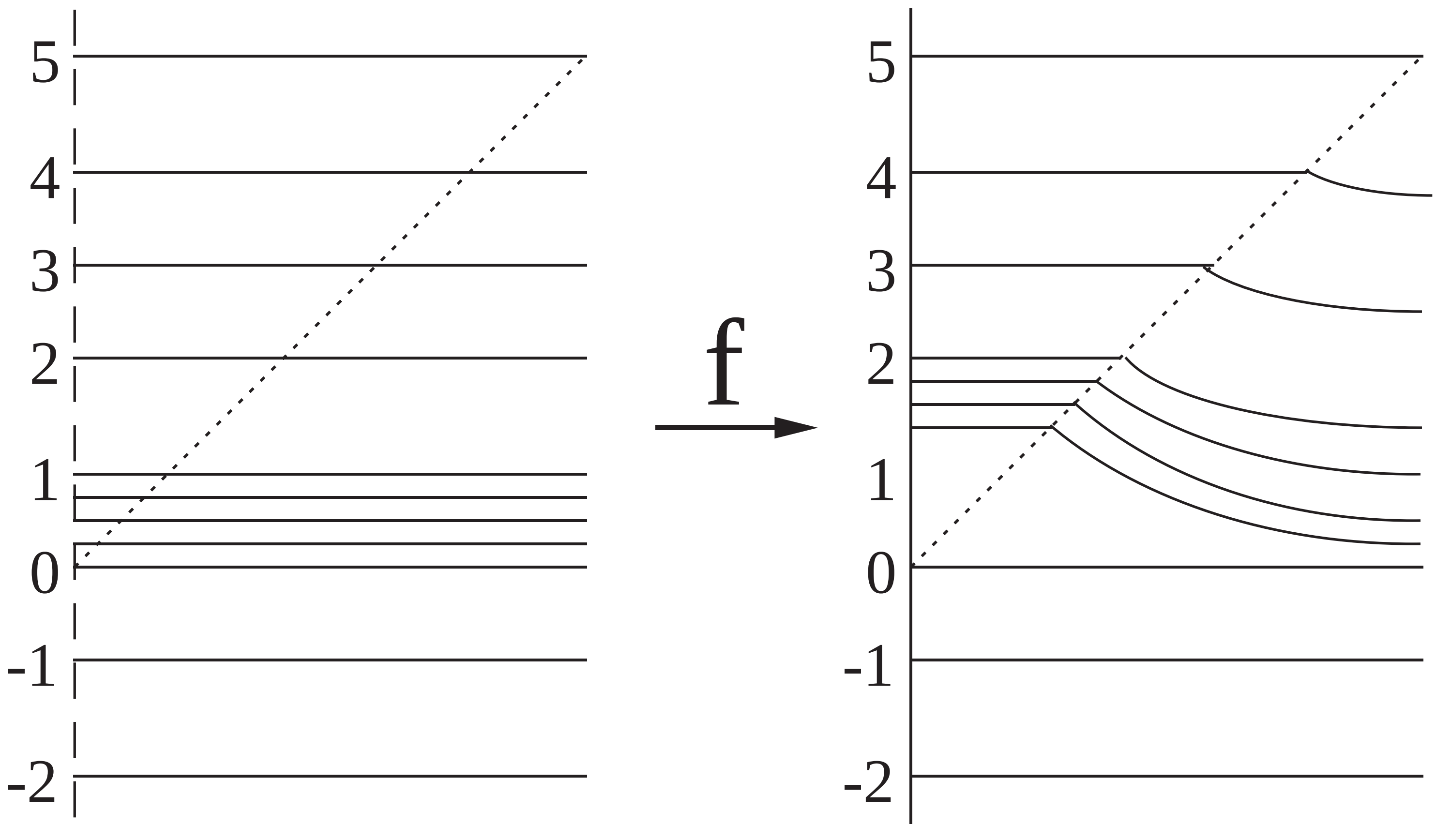

P_0^+



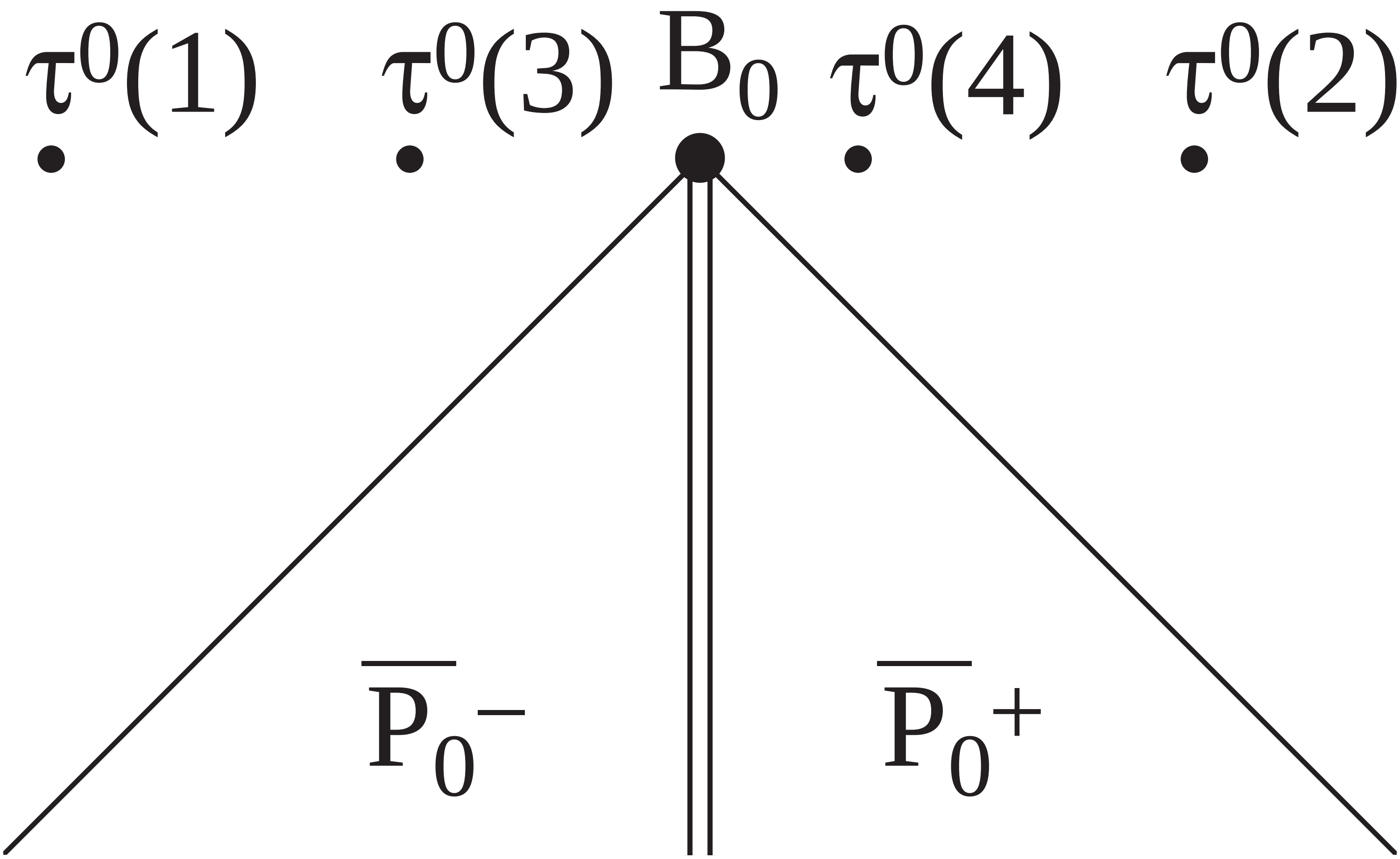
w
U
x_{n_k}
x
...
γ
γ_k
q
p
z

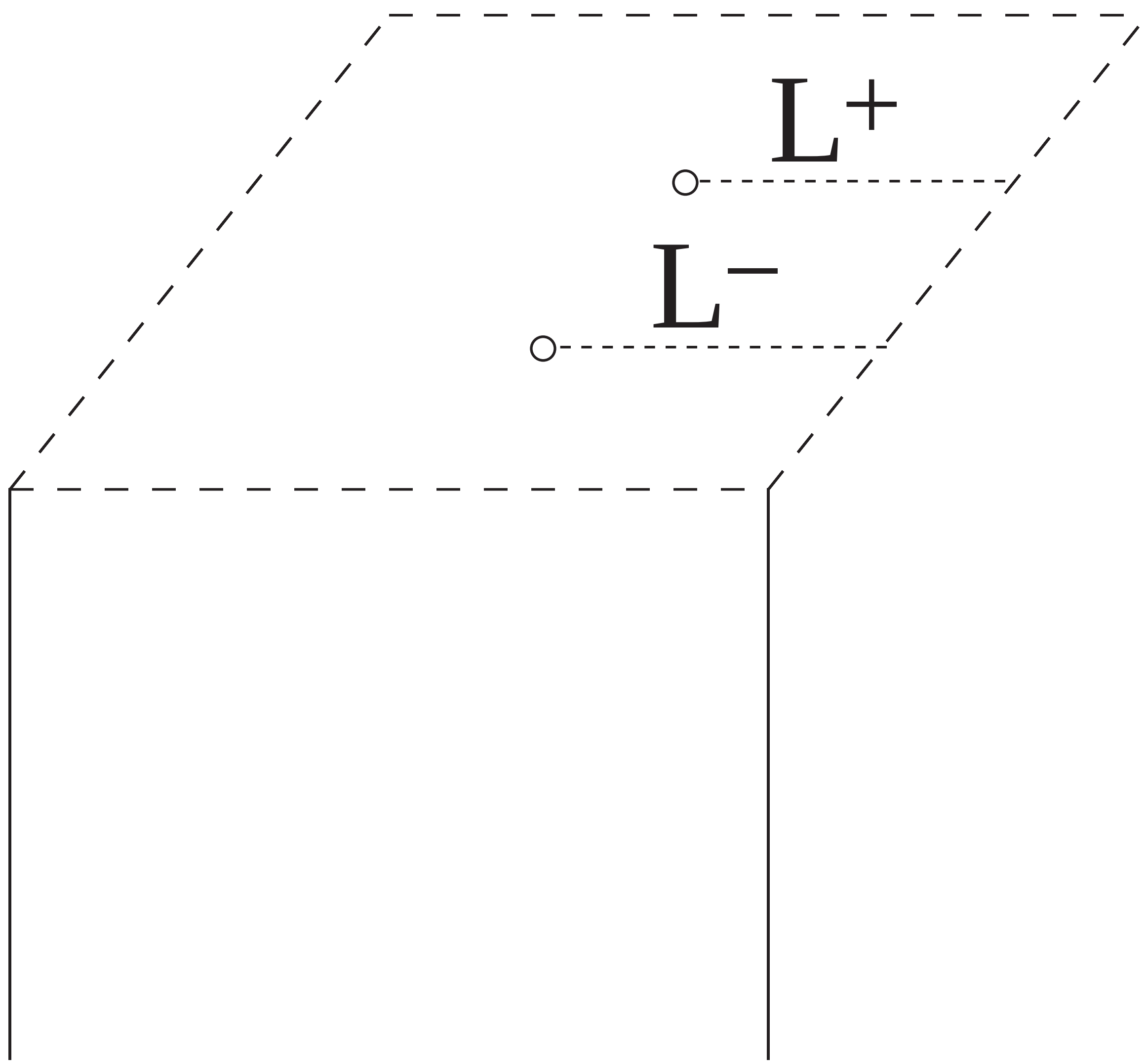


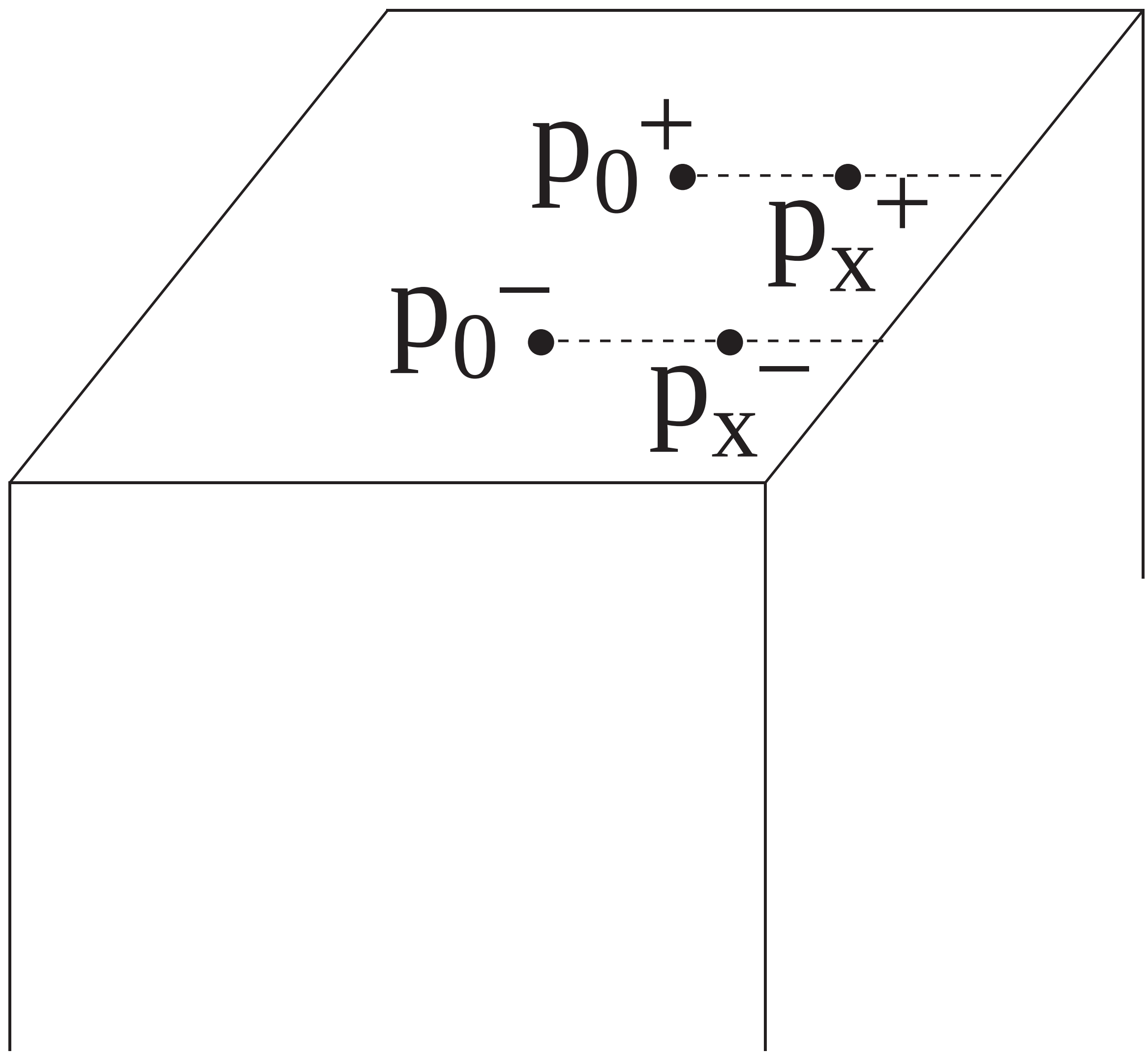
x_{n_k}
y
y^n
$y_k{}^n$
c
U
c^n
x
$c_k{}^n$
p
z_n



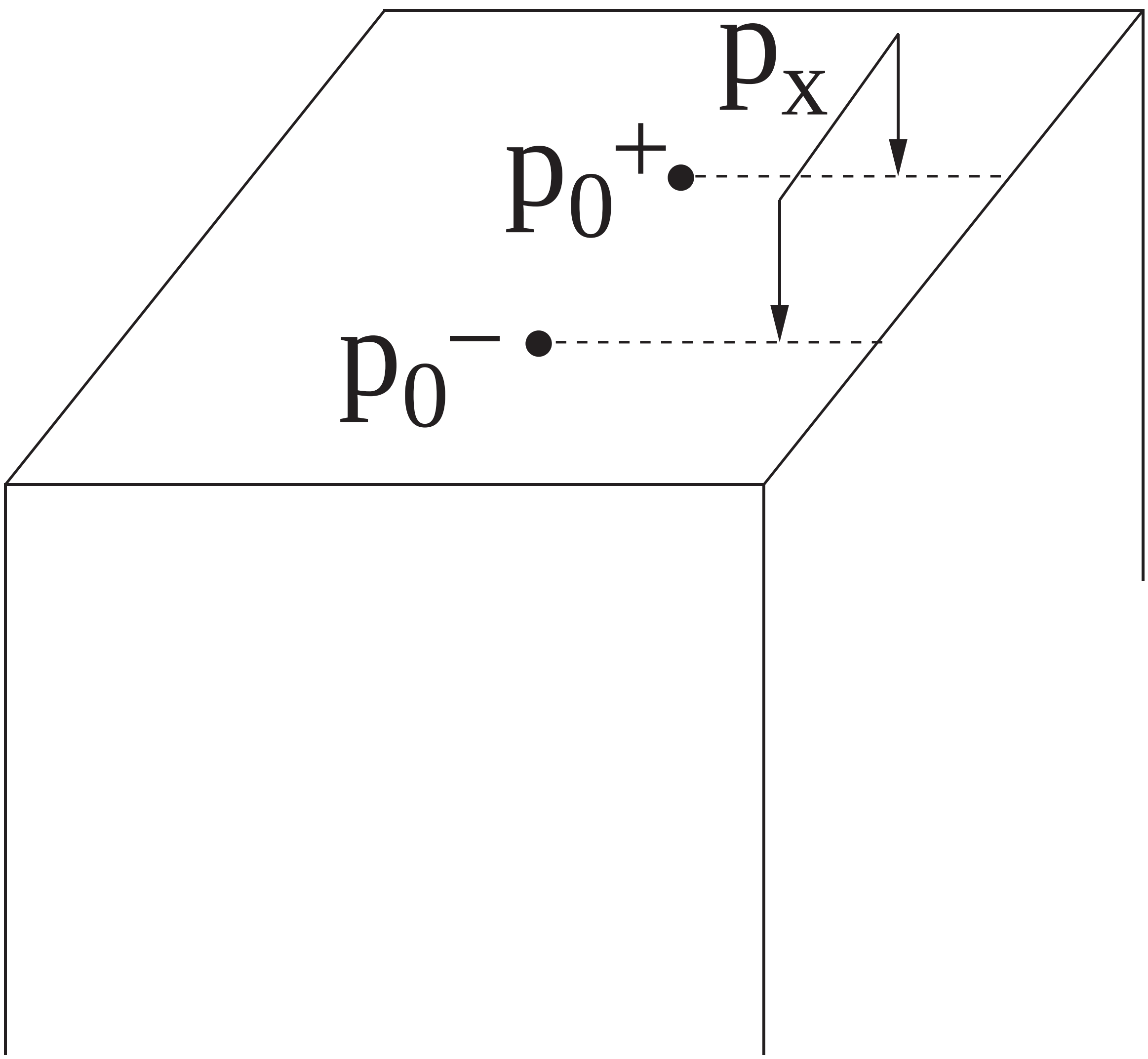
5
4
3
2
1
0
-1
-2
f
5
4
3
2
1
0
-1
-2

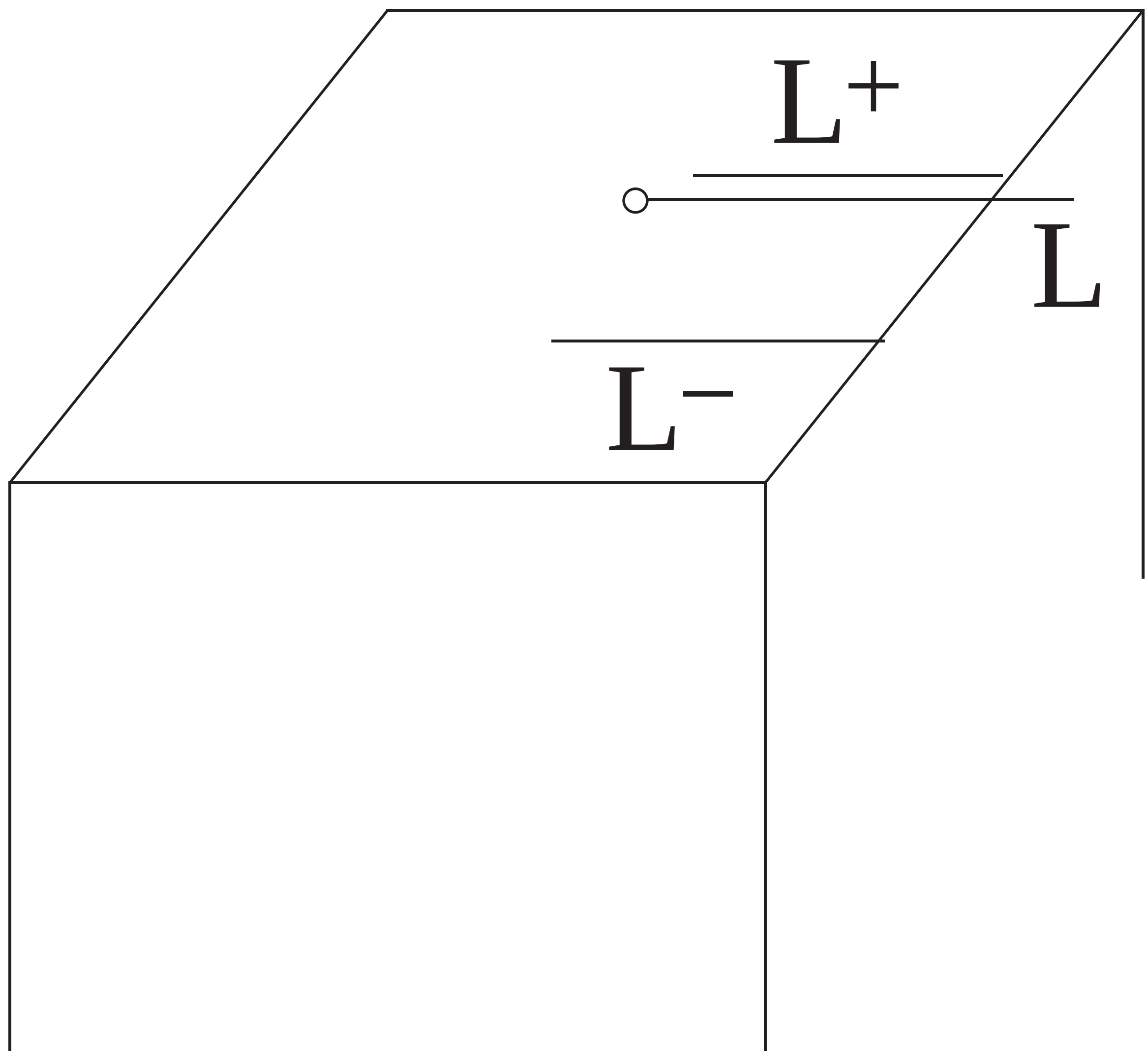

$\tau^0(1)$
$\tau^0(3)$
B_0
$\tau^0(4)$
$\tau^0(2)$
$\overline{P_0}^-$
$\overline{P_0}^+$


L^+
L^-

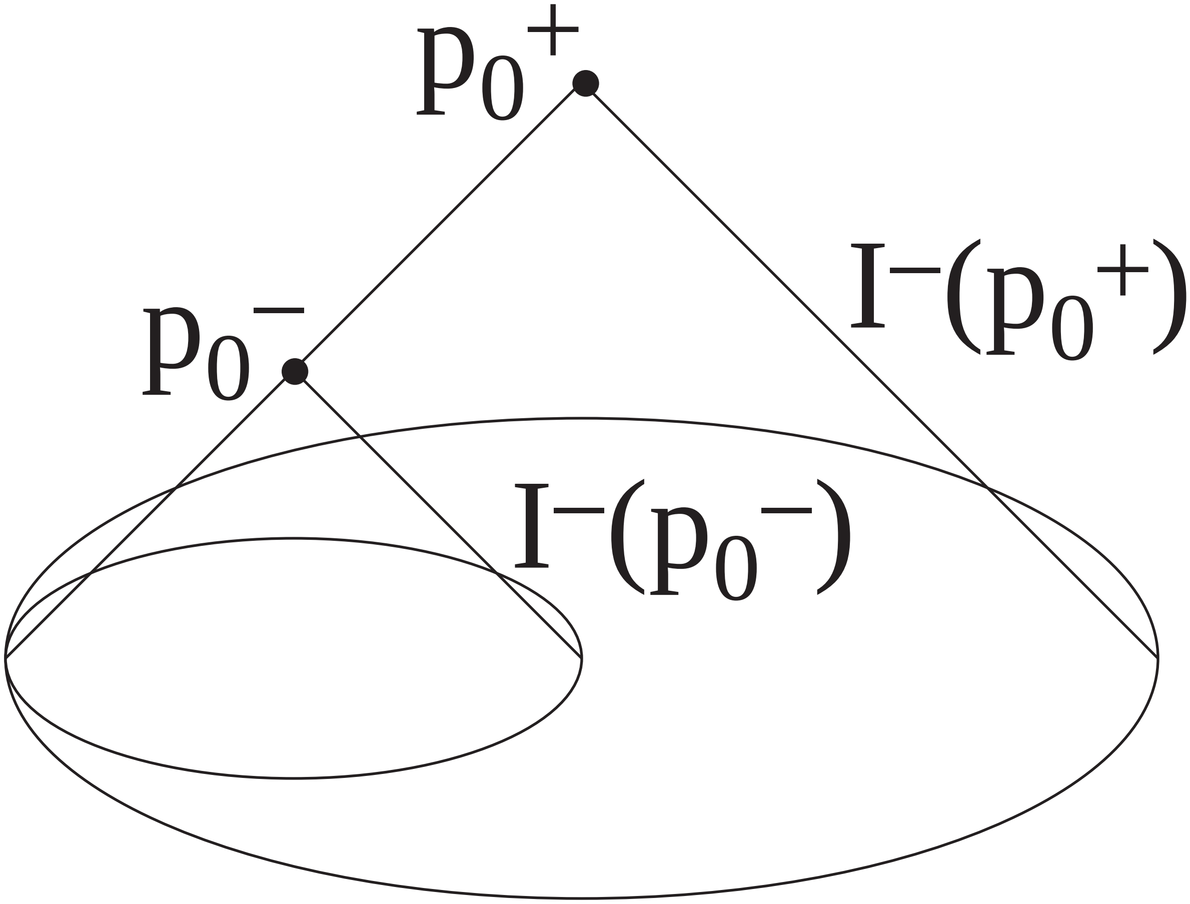


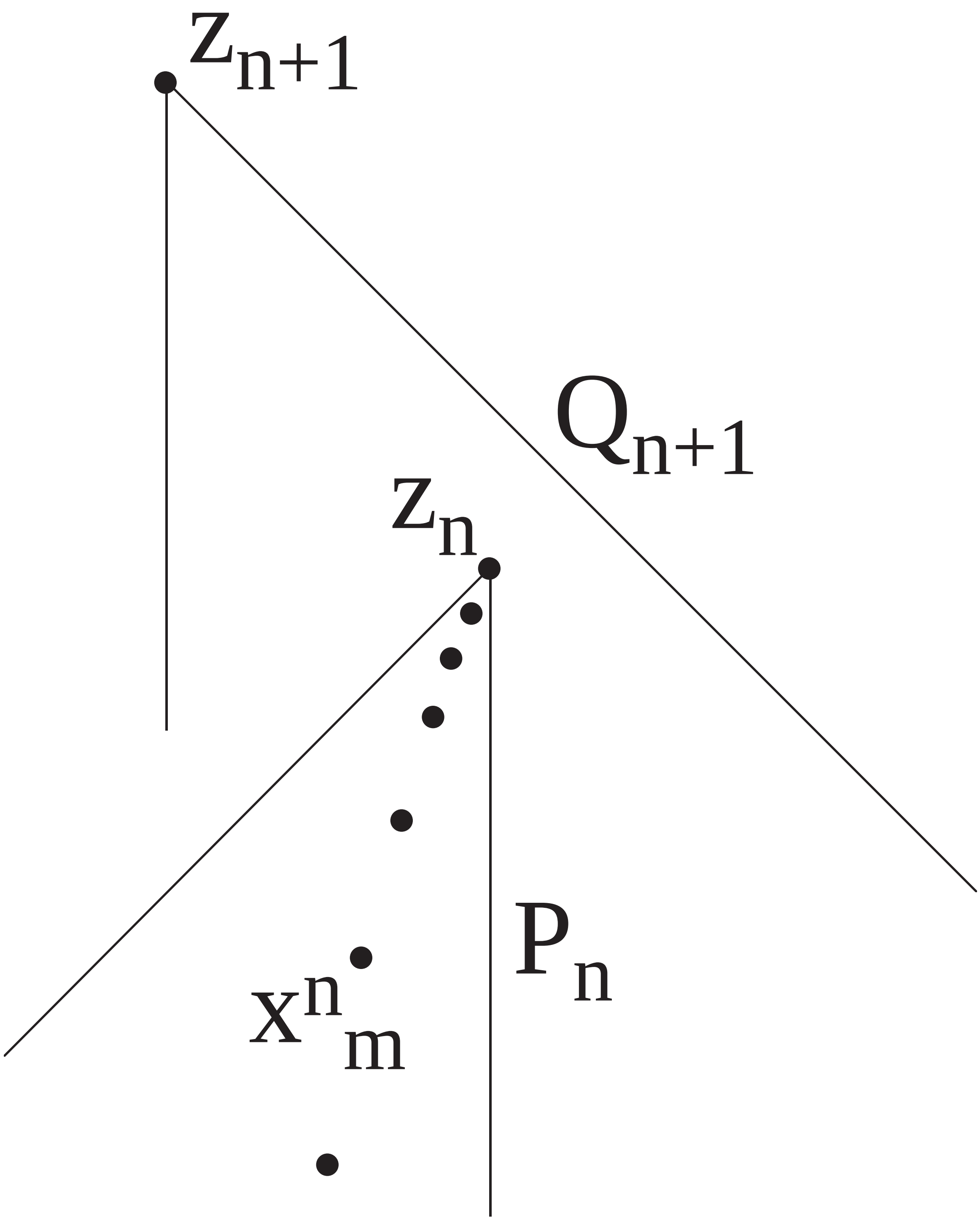
p_0^+
p_x^+
p_0^-
p_x^-


p_x
p_0^+
p_0^-

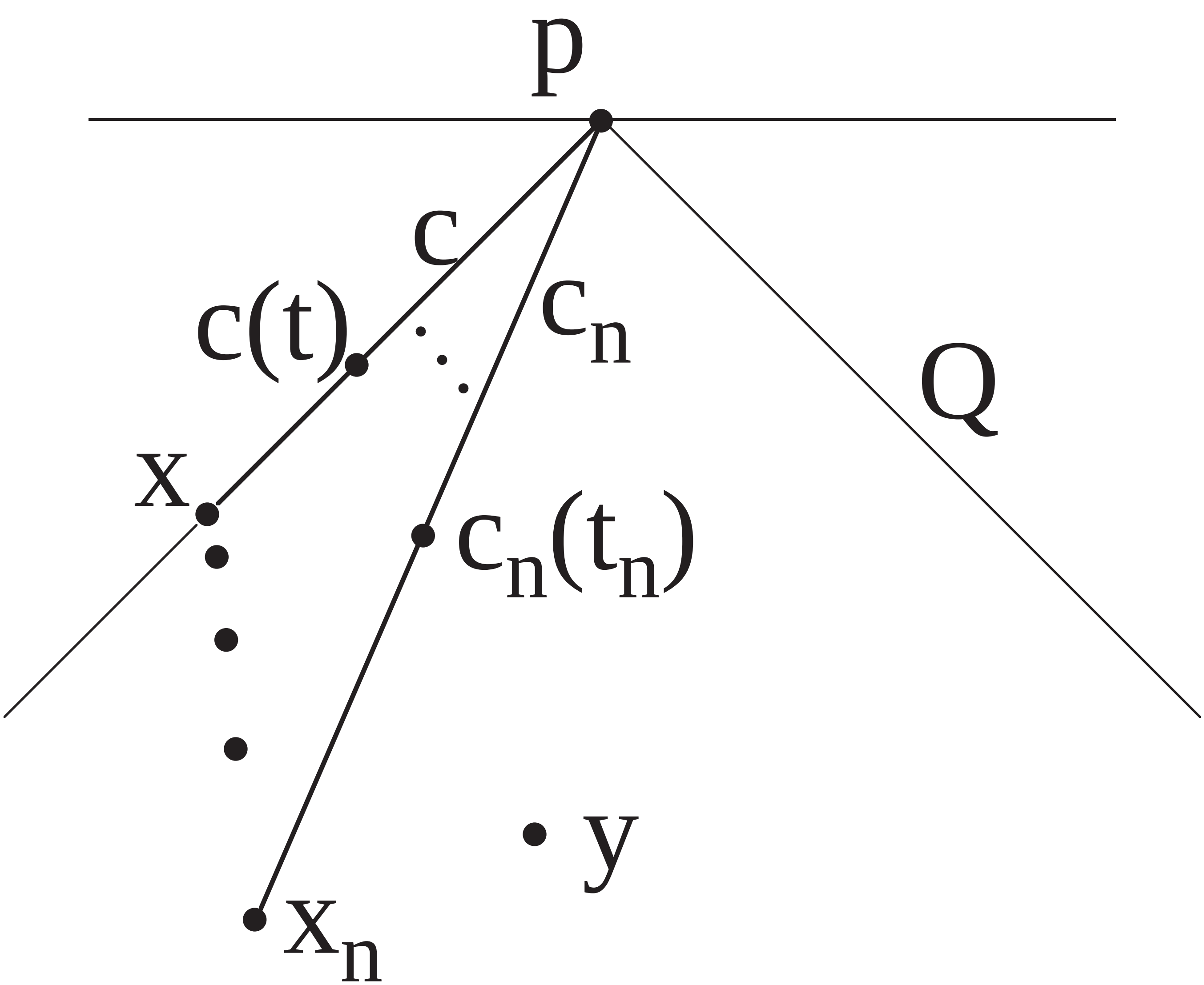


L^+
L
L^-


p_0^+
p_0^-
$I^-(p_0^+)$
$I^-(p_0^-)$



z_{n+1}
Q_{n+1}
z_n
P_n
x^n_m


p
c
c(t)
c_n
Q
x
$c_n(t_n)$
y
x_n

TOPOLOGY OF THE FUTURE CHRONOLOGICAL BOUNDARY: UNIVERSALITY FOR SPACELIKE BOUNDARIES

Steven G. Harris

Abstract. A method is presented for imputing a topology for any chronological set, i.e., a set with a chronology relation, such as a spacetime or a spacetime with some sort of boundary. This topology is shown to have several good properties, such as replicating the manifold topology for a spacetime and replicating the expected topology for some simple examples of spacetime-with-boundary; it also allows for a complete categorical characterization, in topological categories, of the Future Causal Boundary construction of Geroch, Kronheimer, and Penrose, showing that construction to have a universal property for future-completing chronological sets with spacelike boundaries. Rigidity results are given for any reasonable future completion of a spacetime, in terms of the GKP boundary: In the imputed topology, any such boundary must be homeomorphic to the GKP boundary (if all points have indecomposable pasts) or to a topological quotient of a closely related boundary (if boundaries are spacelike). A large class of warped-product-type spacetimes with spacelike boundaries is examined, calculating the GKP and other possible boundaries, and showing that the imputed topology gives expected results; included among these are the Schwarzschild singularity and those Robertson-Walker singularities which are spacelike.

1. The Future Chronological Boundary

1.1 Introduction.

In 1973, Geroch, Kronheimer, and Penrose introduced in [GKP] the notion of the Causal Boundary for a strongly causal spacetime. This is a seemingly very natural method of appending a future endpoint to each future-endless timelike curve in a spacetime (and a past endpoint to each past-endless timelike curve), in a manner which is conformally invariant and which depends, for the future endpoints, only on the past of each curve (and dually for the past endpoints). If M denotes the spacetime and $\partial^{\#}(M)$ its Causal Boundary, then we speak of $M^{\#} = M \cup \partial^{\#}(M)$ as the "completion" of M by the causal boundary. This comes equipped with a chronology relation $\ll^{\#}$ and a causality relation $\prec^{\#}$ which extend those ($\ll$ and $\prec$) on M; however, $x \ll^{\#} y$ or $x \prec^{\#} y$ may possibly obtain in $M^{\#}$ for x and y in M, even though $x \ll y$ or $x \prec y$ do not hold (though this cannot happen if M is globally hyperbolic). There is also defined a topology on $M^{\#}$ which induces the original topology on M as a subspace and makes the elements of $\partial^{\#}(M)$ actual endpoints to the appropriate curves in M.

For those interested in understanding the large-scale structure of spacetimes in terms of a natural boundary, this all sounds quite good; but there are problems: The construction of the Causal Boundary consists of first defining what may be called the Future Causal Boundary (the future endpoints of timelike curves) and, separately,

the Past Causal Boundary. Then these two are melded together in an elaborate procedure (though this is quite simple if there are only spacelike components of the boundary and no naked singularities). A very complicated topology is defined on this combined boundary, which is further adumbrated with quotients so as to make it Hausdorff. The total complications are quite formidable, and very little has been done in the way of computing the Causal Boundary for specific spacetimes (an example of an explicit calculation for a flat, multiply-connected two-dimensional spacetime is given in [HD]; this also gives an example of how the Causal Boundary may be utilized to help understand the behavior of a spacetime). Besides this, the topology as defined by GKP is not, actually, at all what one might expect; for instance, for Minkowski space, the Causal Boundary is a pair of null cones (one each for the future and the past), just as in the standard conformal embedding of Minkowski space into the Einstein static universe (see [HE])—except that each cone element (null line) is an open set!

But, still, the notion of the Causal Boundary seems, somehow, very natural. Can that be established in a rigorous sense, and, if so, can information about other boundaries of spacetimes be derived from knowledge of the Causal Boundary? It is the purpose of an intended series of articles, of which this is the second, to give affirmative answers to these questions.

The first article, [H], established a rigorous sort of "naturalness" for a portion of the Causal Boundary: the Future Causal Boundary (without melding with the Past Causal Boundary), with its chronology relation only (no causality relation and no topology). To emphasize that this is only a portion of the full GKP Causal Boundary, which has both chronology and causality relations, it was called the Future Chronological Boundary, and that nomenclature is preserved here. The form of "naturalness" explicated in that article (and this) is categorical: showing that the constructions defined are functorial, natural, and universal, in the sense of category theory. The benefit derived from this is that one then is assured that the constructions are categorically unique for having the requisite properties—any other functorial construction with the same naturalness and universality must be naturally equivalent to the one at hand, in the strict categorical sense.

Here is what all that means in simplified outline: First there was defined a category of objects with morphisms between those objects, a category large enough to contain both spacetimes and spacetimes-with-boundaries (and chronology-preserving continuous functions between spacetimes); these objects are called chronological sets, and the morphisms, future-continuous functions. Then it was shown that adding the Future Chronological Boundary $\partial^+(X)$ to a chronological set X produces a future-complete chronological set X^+; and that this process is a functor from the category of chronological sets to the subcategory of future-complete chronological sets (so for any future-continuous $f : X \to Y$ between chronological sets, we obtain the extension $f^+ : X^+ \to Y^+$ of f). The inclusion $\iota^+_X : X \to X^+$ was shown to be a natural transformation (i.e, it commutes with f and f^+: $f^+ \circ \iota^+_X = \iota^+_Y \circ f$). Finally, the appropriate universality property was established: For any future-continuous $f : X \to Y$ for which Y is already future-complete, f^+ is the unique future-continuous extension of f to X^+, i.e., so that $f^+ \circ \iota^+_X = f$. The upshot is that the future completion process is thus left adjoint to the forgetful functor from future-complete chronological sets to chronological sets; and left adjoints, by category theory, are unique up to natural equivalence (natural transformations consisting of isomorphisms)—see [M]. It is the universality principle— $f^+ \circ \iota^+_X = f$

that is the most useful piece of information, allowing comparison of any other way of future-completing X (i.e., of mapping X into any future-complete object Y) with the GKP future completion (i.e., X^+).

It must be emphasized that despite the usage of the phrase "future-continuous function", there was no real topology in the constructions above: The term "future-continuous" refers to preservation of "future limits", a generalization of future endpoints for timelike curves; there was no information in those constructions relating to more general notions of convergence. It is the purpose of this paper to fill that lacuna: to establish functoriality, naturalness, and universality for the Future Chronological Boundary construction, including appropriate topology—at least in the case of spacelike boundaries.

One of the reasons why categorical properties such as universality are important for a boundary construction process, is that they allow one to deduce much about all possible boundaries—at least, those meeting the same basic conditions, such as being completing objects in the same category. As a consequence, there are developed here several important rigidity (or quasi-rigidity) results for future-completing boundaries on spacetimes; these take the form of saying that any "reasonable" future-completing boundary on a spacetime must be topologically identical to (or, depending on the hypotheses, a topological quotient of) the GKP Future Causal Boundary—and the GKP boundary is fully explicated here for a number of classical spacetimes, such as interior Schwarzschild and some Robertson-Walker spaces. However, these results depend crucially on accepting a specific topology for the chronological sets (spacetime *cum* boundary) that one is looking at; and a large burden of this paper is to make the case for the acceptance of this topology.

The construction of X^+ in [H] was divided into two procedures, each independently functorial, natural, and universal: First, the construction of the Future Causal Boundary $\hat{\partial}(X)$ of a chronological set X, with a "first approximation" of the needed chronology relation on $\hat{X} = X \cup \hat{\partial}(X)$ (i.e., a relation, $\hat{\ll}$, that extends that of $\ll$ on X without relating any elements of X that $\ll$ does not); this is called the future completion functor. Second, a way of extending a chronology relation to relate additional elements in a chronological set which is necessary for certain purposes; this is called the past-determination functor. It is, in essence, the composition of these two functors that results in X^+, the addition to X of what amounts to the GKP Future Causal Boundary.

The second part of Section 1 gives an overview of the results presented here in general form, with some discussion of significance. The third part gives a detailed summary of the constructions and conventions previously established in [H], upon which the rest of this paper builds. A rigidity theorem for "nice" boundaries (in the chronological category) is added in the fourth part.

The basic ideas for the Future Chronological Boundary are contained in the future completion functor, and it is the inculcation of topology with that functor that will be discussed first, in Section 2. However, it must be noted that the future completion functor operates only on chronological sets that are already past-determined. Section 3 explores how topology works with the past-determination functor. Both these Sections assume "regularity": that every point in the chronological sets under discussion has an indecomposable past (true for spacetimes and for some spacetimes-with-boundary). Section 3 ends with important rigidity theorems for regular chronological sets.

But there are spacetimes for which the Causal Boundary construction yields non-regular points: elements of the boundary whose pasts are not indecomposable past sets (typically for boundaries which are timelike and two-sided in the spacetime). This presents a considerable complication for the introduction of topology by the means outlined in Section 2, but it is important to cover such boundaries, as they are sometimes the most natural ones. This forms the subject of Section 4, which concludes with quasi-rigidity theorems covering chronological sets that are not necessarily regular, but have spacelike boundaries.

Finally, Section 5 looks at examples, including some classical spacetimes. Included is a rigidity result for boundaries formed by embedding into larger manifolds.

1.2 Discussion and overview.

The import of the results in [H] was two-fold: First, to show how the GKP construction of the (Future) Causal Boundary is a formally natural and universal construction, in an appropriately (albeit limited) categorical sense; and second, to show how causal and set-theoretic information can be derived about any purported (and "reasonable") future boundary of a spacetime, via a map from the GKP boundary. The idea followed here is to recast these results in a topological frame, by this means:

There will be defined a way to impute a topology for any chronological set X; then the results from [H] will be shown to hold in appropriate topological categories: For instance, morphisms in the new categories will be required to be continuous with respect to the topologies inferred from the chronology relations, and then it will be shown that the same functors as from [H], applied to continuous morphisms, result in continuous morphisms. That will allow the categorical results from [H] to stand in the new topological categories, as well.

But an important part of the problem is to ensure that one has the right topology. What is required for this? At a minimum, we certainly need that if the chronological set $(X, \ll)$ under discussion is actually a spacetime, then the topology inferred from $\ll$ had better be the original manifold topology; the topology defined here (called the $\widehat{\ }$-topology) satisfies this basic requirement.

Beyond that, one may wish to require that if $(X, \ll)$ is a spacetime with some "natural" topological boundary, coming equipped with an extension of the spacetime chronology relation, then the inferred $\widehat{\ }$-topology ought to match this natural topology. It will be shown that this is the case for a class of warped-product spacetimes with "obvious" spacelike boundary, and also with some modifications of that boundary; and that this generalizes to similar modifications of general spacetimes with spacelike boundary.

The spacetimes considered for examples are generalizations of interior Schwarzschild, with its spacelike singularity as the boundary: a product of a portion of the Lorentzian line with a product of complete Riemannian factors (K_i, h_i), $M = (a, b) \times \prod_i K_i$, with a warped product metric $-(dt)^2 + \sum_i f_i(t) h_i$; if the warping functions f_i satisfy an integral condition, then the Future Chronological Boundary of M will be spacelike. (Examples of classical spacetimes with this conformal structure are interior Schwarzschild, the Kasner spacetimes, and many standard static spacetimes.) One anticipates that the boundary of M "ought" to be $K = \prod_i K_i$, attached to M so as to form $\bar{M} \cong (a, b] \times K$; and this is precisely what $M^+ = \hat{M}$ is in the $\widehat{\ }$-topology. This is some confirmation that the inferred topology

is doing what it ought to do.

Our anticipation of that form for the boundary of M comes from a canonical embedding of M into $\mathbb{R} \times K$. But one can imagine many other embeddings $\phi : M \to N$, yielding some construct $\bar{M}_\phi$ as some sort of topological completion of M. There is also a natural chronology relation on $\bar{M}_\phi$, so we can ask if the $\widehat{\ }$-topology from that chronology relation replicates the natural topology on $\bar{M}_\phi$ coming from the embedding ϕ. So long as ϕ extends continuously to $\hat{\partial}(M)$, and that extension is proper onto its image, the answer is yes. Furthermore, this is not restricted to the warped product spacetimes of the previous paragraph, but to any strongly causal spacetime with $\hat{\partial}(M)$ spacelike (one may speak of ϕ extending either to M^+ or to $\hat{M}$, as these are naturally homeomorphic in the $\widehat{\ }$-topologies; past-determination makes only minor changes in the chronology relation, not affecting the $\widehat{\ }$-topology).

The insistence on a spacelike boundary for these examples is a crucial element in the exposition featured here; in fact, it is only spacelike boundaries that can be admitted in the category for which the chronological categorical results can be extended to topological ones. Timelike boundaries have intrinsic problems for the categorical extension: There are simple examples of a spacetime M with timelike boundary (such as the $x > 0$ portion of Minkowski 2-space) and continuous, future-continuous map $f : M \to N$, such that the extension of f to the boundary of M (i.e., $\hat{f}$, the application of the future completion functor to f) is discontinuous; indeed, there is no continuous extension of f to the boundary. But future-continuity of f prevents this happening for spacelike boundaries (what happens for null boundaries is an open question at this time). Thus, the topological category featured will be chronological sets X with $\hat{\partial}(X)$ spacelike.

An important reason for believing in the correctness of the topology construction featured here—the $\widehat{\ }$-topology for any chronological set—is the fact that it does lead to strongly categorical results. The fullest statement of these results is Theorem 3.4, applicable in the category of past-distinguishing, regular chronological sets with spacelike boundaries. The truly interesting results—the ones with the most clear physical impact—are the rigidity theorems which, utilizing the $\widehat{\ }$-topology, sharply curtail the allowable boundaries on spacetimes. But these results are interesting only in so far as one believes that the $\widehat{\ }$-topology is the way to understand the topological behavior of the boundaries in question when defined only by their chronological characteristics. Thus, in a somewhat curious fashion, the categorical results both help establish the *bona fides* of the $\widehat{\ }$-topology construction, and also yield some of the more interesting topological consequences in terms of that $\widehat{\ }$-topology.

We then have these imports for the topological results: First, the GKP construction, with topology, is seen to be the categorically unique way to future-complete a spacetime in a manner which has a universal relation to any other future completion (at least, with respect to spacetimes with spacelike boundaries). Second, that universality property gives us a way of relating—as a topological quotient—any proposed future boundary to the GKP boundary (in case that boundary is spacelike); this is a sort of quasi-rigidity for future boundaries. Third, that universality (and associated quasi-rigidity) is applicable in cases beyond the strictly categorical ones, including situations involving "non-regular" spaces, where the past of a boundary point may not be indecomposable (these must be accounted as reasonable boundary constructions). Fourth, for regular boundaries, there is a strong rigidity result

applicable even without the assumption of spacelike boundaries.

The most important results of this paper probably are the rigidity results of Sections 3, 4, and 5:

Theorem 3.6 (foreshadowed by Theorem 1.1) has this implication: For any spacetime M, any regular, past-distinguishing future completion of M must be homeomorphic (in the $\widehat{\ }$-topology) to $\hat{M}$, and the future-completing boundary must be similarly homeomorphic to the Future Chronological Boundary. (Note that this result is not restricted to spacelike boundaries.) As an example: Proposition 5.2, applied to interior Schwarzschild, shows that the Schwarzschild singularity, in its GKP formulation, has the topology of $\mathbb{R}^1 \times \mathbb{S}^2$, and Theorem 3.6 asserts that this is the only possible topology for a regular, past-distinguishing, future-completing boundary on that spacetime (assuming the $\widehat{\ }$-topology is used for such a boundary).

Theorem 4.8 addresses the same question with respect to allowing for non-regular, "generalized" past-distinguishing, "generalized" future completions (where "generalized" accommodates for non-regular points): Restricting to the situation of spacelike boundaries, this theorem implies that the completing object must (in the $\widehat{\ }$-topology) be a topological quotient of $\hat{M}$; from Corollary 4.9, the completing boundary must be a topological quotient of what I call the Generalized Future Chronological Boundary (a subset of the Future Chronological Boundary, removing IPs corresponding to non-regular points). Applied to the Schwarzschild singularity, this means that anything even approaching a reasonable future completing object can be only a quotient of $\mathbb{R}^1 \times \mathbb{S}^2$—again, using the $\widehat{\ }$-topology.

Theorem 5.3, showing that embeddings (of a proper sort) yield future completions with natural topology the same as the $\widehat{\ }$-topology, then yields a quasi-rigidity for this most common means of completing a spacetime. For Schwarzschild, this means that if we attempt to derive a topology for the singularity, not by imposing the $\widehat{\ }$-topology on a boundary with some chronology relation, but by the purely topological means of embedding the spacetime into a larger manifold, then the same result still holds (if the embedding is proper onto its image): The only possibilities for the singularity are quotients of $\mathbb{R}^1 \times \mathbb{S}^2$.

1.3 Summary of previous nomenclature and constructions.

The following is all from [H]:

A *chronological set* is a set X together with a relation $\ll$ which is transitive and non-reflexive, such that every point in X is related to at least one other point, and such that there is a countable subset S of X so that for all $x \ll y$ in X, there is some $s \in S$ with $x \ll s \ll y$. For any $x \in X$, the *past* of x is $I^-(x) = \{y \mid y \ll x\}$; for $A \subset X$, the past of A is $I^-[A] = \bigcup_{a \in A} I^-(a)$. (The use of square brackets for a subset is to be noted, as subsets of a chronological set X will often be used as elements of an extension of X, and it is needful to distinguish whether one is speaking of the past of a subset *qua* subset of X or *qua* element of the extension; in general, parentheses denote the application of a function to an element of its proper domain, while brackets denote the usage of that function as an operator on subsets of the function's domain.) Futures are defined dually, using I^+. A chronological set X is called *past-distinguishing* if $I^-(x) = I^-(y)$ implies $x = y$. Any strongly causal spacetime is a past-distinguishing chronological set (Theorem 4 in [H]).

A *future chain* in a chronological set X is a sequence $\{x_n\}$ of elements of X obeying $x_n \ll x_{n+1}$ for all n; in a chronological set, future chains serve in the role of timelike curves in a spacetime. A point x is a *future limit* of a future chain c

if $I^-(x) = I^-[c]$; if X is past-distinguishing, then a future chain can have at most one future limit. A function $f : X \to Y$ between chronological sets is *chronological* if $x \ll y$ implies $f(x) \ll f(y)$; note that this implies a future chain gets mapped to a future chain. A chronological function is *future-continuous* if it preserves future limits of future chains. For strongly causal spacetimes, a future limit of a future chain is precisely the same as a topological limit of the sequence; functions which are both past- and future-continuous are the same as continuous functions which preserve future-directed timelike curves (Theorem 4 in [H]).

A non-empty subset $P \subset X$ is a *past set* if $I^-[P] = P$; a past set P is *indecomposable* if it cannot be written as the union of proper subsets which are past sets. For each indecomposable past set (or IP) P there is a (non-unique) future chain c such that $P = I^-[c]$ (c is said to *generate* P), and any set of the form $I^-[c]$, for c a future chain, is an IP (Theorem 3 in [H]). This is in strict analogy with IPs in strongly causal spacetimes being precisely those subsets which are the pasts of timelike curves. A past set P is an IP if and only if for every pair of points in P, there is a third point in P to the future of each of the first two (Theorem 2 in [H]).

A chronological set is called *future-complete* if every future chain has a future limit. No strongly causal spacetime is future-complete. It is the job of the future completion functor to provide a future-complete chronological set $\hat{X}$ for each chronological set X. This is accomplished by adding to the point-set X the *Future Chronological Boundary* of X, $\hat{\partial}(X) = \{P \mid P \text{ is an IP in } X \text{ such that } P \text{ is not } I^-(x) \text{ for any point } x \in X\}$; we let $\hat{X} = X \cup \hat{\partial}(X)$, the *future (chronological) completion* of X. By Theorem 5 in [H], this is a chronological set under the following relation, where x and y are any points in X, P and Q any elements of $\hat{\partial}(X)$:

(1) $x \mathrel{\hat{\ll}} y$ iff $x \ll y$
(2) $x \mathrel{\hat{\ll}} Q$ iff $x \in Q$
(3) $P \mathrel{\hat{\ll}} y$ iff for some $w \in I^-(y)$, $P \subset I^-(w)$
(4) $P \mathrel{\hat{\ll}} Q$ iff for some $w \in Q$, $P \subset I^-(w)$.

To avoid confusion, $\hat{I}^-$ will sometimes be used to denote the past in $\hat{X}$, to distinguish from the past in X.

The future completion of a chronological set is, as the name suggests, future-complete: For any future chain c in X, $I^-[c]$ is an IP which either is $I^-(x)$ for some x which is a future limit for c (in X), or is itself a future limit for c in $\hat{X}$; for a chain c including a subsequence of elements of $\hat{\partial}(X)$, there are interpolated elements of X (as per (3) and (4) above), and these generate the future limit for c. The inclusion map $\hat{\imath}_X : X \to \hat{X}$, the *standard future injection* for X, is future-continuous. If X is past-distinguishing, then so is its future completion. If X is itself future-complete, then $\hat{X} = X$.

Let $f : X \to Y$ be a chronological function between chronological sets with Y past-distinguishing; we need to define an extension of f to the future completions, $\hat{f} : \hat{X} \to \hat{Y}$. For $x \in X$, we define $\hat{f}(x) = f(x)$. For $P \in \hat{\partial}(X)$, generated by a future chain c, consider $Q = I^-[f[c]]$, an IP in Y: Either $Q = I^-(y)$ for a unique $y \in Y$, in which case we define $\hat{f}(P) = y$; or there is no such point in Y, in which case $Q \in \hat{\partial}(Y)$, and we define $\hat{f}(P) = Q$.

In order that $\hat{f}$ also be chronological, we need an additional assumption on Y: We define a chronological set to be *past-determined* if whenever $I^-(y) \subset I^-(w)$ and $w \ll z$, we also have $y \ll z$ (mimicking the definition of $\hat{\ll}$ for future comple-

tions). This is true for globally hyperbolic spacetimes, but false for many spacetimes with “holes” (such as Minkowski 2-space with a spacelike half-line removed). If a chronological set is past-determined, then so is its future completion.

Now let $f : X \to Y$ be a chronological map with Y past-distinguishing and past-determined; then $\hat{f}$ is also chronological. Moreover, if f is future continuous then $\hat{f}$ is the unique future-continuous map satisfying $\hat{f} \circ \hat{\iota}_X = \hat{\iota}_Y \circ f$. An alternative formulation: For $f : X \to Y$ future-continuous with Y future-complete as well as past-determined and past-distinguishing, $\hat{f} : \hat{X} \to Y$ is the unique future-continuous function with $\hat{f} \circ \hat{\iota}_X = f$ (Proposition 6 in [H]); this is the universality property.

This last allows us to define things categorically: Let **PdetPdisChron** be the category of past-determined, past-distinguishing chronological sets with future-continuous functions as the morphisms, and let **FcplPdetPdisChron** be the subcategory with future-complete objects (and the same morphisms). We then have that future completion is a functor $\hat{\ }$: **PdetPdisChron** $\to$ **FcplPdetPdisChron** (that $\widehat{g \circ f} = \hat{g} \circ \hat{f}$ follows from the uniqueness of extending a function to the future completions, since $\hat{g} \circ \hat{f}$ has the requisite properties for the extension of $g \circ f$). The standard future injections $\hat{\iota}_X$ yield a natural transformation $\boldsymbol{\iota}$ from the identity functor on **PdetPdisChron** to $\hat{\ }$, and $\hat{\ }$ is left-adjoint to the “forgetful” (i.e., inclusion) functor from **FcplPdetPdisChron** to **PdetPdisChron** (that just means that the universality property above holds). This is important, since left-adjoints are categorically unique (i.e., unique up to natural equivalence): future completion is the categorically unique way to create a future-complete chronological set from a given past-determined, past-distinguishing one.

A substantial awkwardness is that future completion requires a past-determined object if it is to act functorially: If the target of a future-continuous function f is not past-determined, $\hat{f}$ may very well not be chronological, and many spacetimes are not past-determined. The remedy is the past-determination functor, which (categorically) extends the chronology relation to additional pairs of points. Specifically: For any chronological set X with chronology relation $\ll$, the *past-determination* of X, written X^p, is X with the chronology relation $\ll^p$, defined by $x \ll^p y$ if $x \ll y$ or if $I^-(x)$ is non-empty and for some $w \ll y$, $I^-(x) \subset I^-(y)$; I^{-p} will be used to denote the past in X^p, as necessary. X^p is the *past-determination* of X; it is past-determined, and if X is already past-determined, then $X^p = X$. If X is, respectively, past-distinguishing or future-complete, then so is X^p (Proposition 10 in [H]). The function $\iota^p_X : X \to X^p$ which, on the set level, is the identity, is future-continuous.

For any future chain c in X^p (i.e., a chain with respect to the relation $\ll^p$) there is a (non-unique) future chain c' in X, said to be *associated* to c, such that a point x is a future limit of c in X^p if and only if it is a future limit of c' in X (Proposition 11 in [H]); for instance, for $\cdots x_n \ll^p x_{n+1} \cdots$, we have $I^-(x_n) \subset I^-(w_n)$ for some $w_n \ll x_{n+1}$; then $\{w_n\}$ is an associated chain in X.

To be properly functorial, past-determination requires just a bit more in the way of hypothesis: In [H] a chronological set X was called *past-connected* if every point is a future limit of some future chain; this includes strongly causal spacetimes. As mentioned in the beginning of Section 2 below, a better nomenclature is this: Call a point $x \in X$ *regular* if $I^-(x)$ is indecomposable; and call X regular if all its points are regular (actually “past-regular” would be more appropriate, from the

standpoint of time-duality). This is equivalent to being past-connected: For x to be the future limit of a future chain is precisely to say that $I^-(x) = I^-[c]$ for some future chain c, and that is equivalent to $I^-(x)$ being indecomposable. The notion of this being an ordinary, every-day sort of behavior to expect from a point is helpful to associate with this concept; thus, what was called past-connected in [H] will be called regular here, with **Preg** substituting for **Pcon** in the naming of categories.

If $f : X \to Y$ is a future continuous function and X is regular, then X^p is still regular and $f^p : X^p \to Y^p$ is future continuous, where f^p is the same set-function as f. Thus we have a functor $\mathbf{p}$: **PregChron** $\to$ **PdetPregChron**, where the infix **-Preg-** denotes "(past-)regular". The maps ι^p_X form a natural transformation $\boldsymbol{\iota}^{\mathbf{p}}$, i.e., for $f : X \to Y$ in **PregChron**, $f^p \circ \iota^p_X = \iota^p_Y \circ f$. Finally, we have the requisite universality property: For $f : X \to Y$ in **PregChron** and Y already past-determined, $f^p : X^p \to Y$ is the unique future-continuous function satisfying $f^p \circ \iota^p_X = f$ (Corollary 12 in [H]). This means the past-determination functor is also a left-adjoint to a forgetful functor (inclusion of **PdetPregChron** in **PregChron**); thus, past-determination is the categorically unique way to create a past-determined chronological set from a given regular one.

These two functors, past-determination and future completion, compose, as do the respective natural transformations, yielding another left-adjoint functor $\widehat{\ } \circ \mathbf{p}$: **PregPdisChron** $\to$ **FcplPdetPregPdisChron**, the categorically unique way to create a future-complete and past-determined chronological set from a given past-distinguishing, regular one. However, this is not quite the GKP Future Causal Boundary: That construction is actually $(\hat{X})^p$ (which we will write as X^+, as above), rather than $\widehat{X^p}$. These two are actually isomorphic via $j_X : \widehat{X^p} \to (\hat{X})^p$, defined by $j_X(x) = x$ for $x \in X$, and, for $P \in \hat{\partial}(X^p)$ generated by a future chain c in X^p, $j_X(P) = I^-[c']$, where c' is any future chain in X associated to c; j_X is future-continuous and has a future-continuous inverse ($(j_X)^{-1}$ maps $I^-[c] \in \hat{\partial}(X)$ to $I^{-p}[c]$) (Proposition 13 in [H]).

We need to have a functor associated with the X^+ construction, but it cannot be done by setting $f^+ = (\hat{f})^p$, because that is not, in general, future-continuous (one needs the target of f to be past-determined in order for $\hat{f}$ to be future-continuous). We finesse this difficulty by defining, for $f : X \to Y$ in **PregPdisChron**, $f^+ = j_Y \circ \widehat{f^p} \circ (j_X)^{-1} : X^+ \to Y^+$; then we have the functor $\mathbf{+}$: **PregPdisChron** $\to$ **FcplPdetPregPdisChron**. The maps $\iota^+_X = j_X \circ \hat{\iota}_{X^p} \circ \iota^p_X : X \to X^+$ define a natural transformation $\boldsymbol{\iota}^+$, and we have the universality property: For any $f : X \to Y$ in **PregPdisChron** with Y future-complete and past-determined, $f^+ : X^+ \to Y$ is the unique future-continuous map satisfying $f^+ \circ \iota^+_X = f$ (Theorem 14 in [H]). Thus, $\mathbf{+}$ is left-adjoint to the same forgetful functor as is $\widehat{\ } \circ \mathbf{p}$, so the two are naturally equivalent (the maps j_X provide the natural equivalence $\mathbf{j} : \widehat{\ } \circ \mathbf{p} \stackrel{\cdot}{\to} \mathbf{+}$).

1.4 A Chronological Rigidity Theorem.

Here is a result that could have been mentioned in [H], as it has to do solely with the chronological category, but was not: In essence, the Future Chronological Boundary is the *only* way to future-complete a regular chronological set.

Suppose we begin with a regular chronological set X and ask how we might define a future completion for it. What we are seeking is a future-complete, past-distinguishing chronological set Y together with a map $i : X \to Y$ such that the restriction $i_0 : X \to i[X]$ is an isomorphism of chronological sets, and such that the remainder of Y—that which is not in $Y_0 = i[X]$—consists solely of future limits

of future chains in Y_0. Then, except for past-determination, Y can only be X^+ and i must be ι^+_X, up to isomorphism (specifically: $\iota^p_y \circ i = i^+ \circ \iota^+_X$, and i^+ is an isomorphism):

Theorem 1.1. *Let X and Y be chronological sets with X regular and Y future-complete and past-distinguishing, and $i : X \to Y$ a future-continuous map obeying*

1. *with $Y_0 = i[X]$, the chronology relation on Y, restricted to Y_0, yields a chronological set;*
2. *with $i_0 : X \to Y_0$ the restriction of i, i_0 is a chronological isomorphism; and*
3. *with $\partial(Y) = Y - Y_0$, every element of $\partial(Y)$ is a future limit of a future chain in Y_0.*

Then $i^+ : X^+ \to Y^+$ is a chronological isomorphism.

Proof. First note that since Y is future-complete, $\hat{Y} = Y$, so $Y^+ = (\hat{Y})^p = Y^p$. Also, since Y is future complete, so is Y^p, so $\widehat{Y^p} = Y^p$. Therefore, $j_Y : \widehat{Y^p} \to (\hat{Y})^p$ is just the identity map on Y^p. Thus, $i^+ = j_Y \circ \widehat{i^p} \circ (j_X)^{-1} = \widehat{i^p} \circ (j_X)^{-1} : (\hat{X})^p \to Y^p$.

Obviously, i^+ is onto Y_0; it is also onto $\partial(Y)$: For any $y \in \partial(Y)$, there is some future chain c in Y_0 with y the future limit of c. Since i_0 is an isomorphism, $c' = i^{-1}[c]$ is a future chain in X. If c' has a future limit $x \in X$, then $i(x) = y$ by future-continuity, so $y \in Y_0$. Therefore, c' has no future limit in X, but it has a future limit $P \in \hat{\partial}(X)$; $\widehat{\iota^p_X}(P)$ is also the future limit of c' in X^p (actually, in $\hat{\partial}(X^p)$). Then $\widehat{i^p}(\widehat{\iota^p_X}(P)) = y$ (it must be the future limit of $i[c'] = c$). Note that $j_X(\widehat{\iota^p_X}(P)) = I^-[c']$; thus, $\widehat{i^p}((j_X)^{-1}(I^-[c'])) = y$.

Clearly, i^+ is injective on X; it is also injective on $\hat{\partial}(X)$: Suppose $i^+(P_1) = y = i^+(P_2)$ for P_1 and P_2 in $\hat{\partial}(X)$. Let P_k be generated by a future chain c_k in X; then $(j_X)^{-1}(P_k)$ is generated by c_k in X^p, hence, is its future limit in X^p. Then, by future-continuity, $\widehat{i^p}((j_X)^{-1}(P_k)) = y$ is the future limit of $i[c_k]$. Thus, $I^-(y) = I^-[\,i[c_1]\,] = I^-[\,i[c_2]\,]$. This means that for each n, there is some m with $i(c_1(n)) \ll i(c_2(m))$, and *vice versa* for 1 and 2 exchanged. Then the same relationship obtains between c_1 and c_2, which means $P_1 = P_2$.

We already know that j_X is an isomorphism of the chronology relation; we must show the same for $\widehat{i^p} : \widehat{X^p} \to Y^p$:

Suppose for x_1 and x_2 in X, $i(x_1) \ll^p i(x_2)$; then there is some $w \ll i(x_2)$ with $I^-(i(x_1)) \subset I^-(w)$. If $w \in \partial(Y)$, we can replace it with $z \in Y_0$ obeying the same relationship: There is some u with $w \ll u \ll i(x_2)$ and if u is not itself in Y_0, then u is the future limit of a chain in Y_0, so there is some $z \in Y_0$ with $w \ll z \ll u \ll i(x_2)$. Thus, we can find $x \in X$ with $I^-(i(x_1)) \subset I^-(i(x))$ and $i(x) \ll i(x_2)$. Then by the isomorphism of i_0, the same relationship obtains in X among x_1, x, and x_2, so $x_1 \ll^p x_2$.

Suppose for $x \in X$ and $y \in \partial(Y)$, $i(x) \ll^p y$. Since y is the future limit of $i[c]$ in Y, it is also the future limit of $i[c]$ in Y^p, *i.e.*, $I^{-p}(y) = I^{-p}[\,i[c]\,]$; that says precisely that $\widehat{i^p}(P) = y$, so we need to show $x \ll^p P$. Let c be a chain in X such that y is the future limit of $i[c]$; for n sufficiently high, $i(x) \ll^p i(c(n))$. By the result above, $x \ll^p c(n)$. Let P be the IP in X generated by c; then we have $x \ll^p c(n) \ll P$ in $\hat{X}$, from which it follows that $x \ll^p P$.

Suppose for $x \in X$ and $y \in \partial(Y)$, $y \ll^p i(x)$. As above, we have $y = \widehat{i^p}(P)$ for P generated by c in X, where y is the future limit of $i[c]$. We can find some $w \in Y$ with $y \ll^p w \ll^p i(x)$, and we can take $w = i(z)$ for some $z \in X$ (if $w \in \partial(Y)$, then

w is the future limit in Y of some chain in Y_0, so it is the future limit in Y^p of such a chain). From $i(c(n)) \ll y \ll^p i(z)$, we derive $i(c(n)) \ll i(z)$, whence $c(n) \ll z$, for all n. From knowing $i(z) \ll^p i(x)$, we learn, by the results above, that $z \ll^p x$, i.e., $I^-(z) \subset I^-(u)$ for some $u \ll x$. Then all $c(n) \ll u$, so $P \subset I^-(u)$. This gives us $P \ll x$, so $P \ll^p x$.

Suppose for y_1 and y_2 in $\partial(Y)$, $y_1 \ll^p y_2$; say $y_k = \widehat{i^p}(P_k)$. As in the paragraph above, we can find $z \in X$ with $y_1 \ll^p i(z) \ll^p y_2$. Then applying the previous two paragraphs shows us that $P_1 \ll^p z \ll^p P_2$, so $P_1 \ll^p P_2$. $\square$

In case Y is regular, there is an equivalent formulation of the hypotheses of this theorem that is worth mentioning: that Y have a subset Y_0 which, under the restriction of the chronology relation from Y, is a chronological set in its own right; that $i[X] = Y_0$ and the restriction $i_0 : X \to Y_0$ is a chronological isomorphism; and that Y_0 is "chronologically dense" in Y, in the sense that for any $y_1 \ll y_2$ in Y, there is some $z \in Y_0$ with $y_1 \ll z \ll y_2$. One direction doesn't require regularity: To find $i(x)$ with $y_1 \ll i(x) \ll y_2$, locate y with $y_1 \ll y \ll y_2$. Either y is already in Y_0 or it is the the future limit of $i[c]$ for some chain $c = \{x_n\}$ in X; in the latter case, for some n, $y_1 \ll i(x_n)$, and $i(x_n) \ll y$. For the other direction: For $y \in Y$, if y is regular, then $I^-(y)$ is generated by a chain $\{y_n\}$, and for each n we pick $z_n \in Y_0$ with $y_n \ll z_n \ll y_{n+1}$.

Theorem 1.1 shows that any past-distinguishing, future-completing boundary on regular X must be isomorphic to $\hat{\partial}(X)$—at least in the past-determinations—both in terms of its own structure and in how it is related to X. This amounts to a rigidity theorem for these sorts of boundaries on regular chronological sets. But this does not mean that there is only one way to put a reasonable future boundary on a spacetime: Important flexibility lies in relaxing insistence on regularity for the completion—in looking, instead, at what were called in [H] generalized future-complete objects. The generalized notions for future limits, future-continuous, future-complete, past-distinguishing, and past-determined, all with applicability to non-regular chronological sets, will be recalled and explored in Section 4.

2. Regular Chronological Sets and Future Completion

In any strongly causal spacetime M, for any point in $x \in M$, $I^-(x)$ is an IP. This is also true for the elements of the Future Chronological Boundary of any chronological set X: If $P \in \hat{\partial}(X)$ is generated by the future chain c (in X), then $\hat{I}^-(P)$ is also generated by c (in $\hat{X}$). The ubiquity of this property is what suggests the nomenclature of "regular" for it, i.e., that a point x is regular if $I^-(x)$ is an IP, and a chronological set X is regular if all its points are regular. Then we have that if X is regular, so are $\hat{X}$ and X^p (if $I^-(x)$ is generated by a future chain c, then c also generates $\hat{I}^-(x)$ and $I^{-p}(x)$).

Non-regular points are typically encountered when combining the Past and Future Causal Boundaries to produce the full Causal Boundary of GKP. A typical example would be with X being Minkowski 2-space $\mathbb{L}^2$ with the negative time-axis $\{(0,t) \mid t \leq 0\}$ removed (see Figure 1): X^+ produces boundary points on either side of the missing semi-axis, $P_s^+ = \{(x,t) \mid t < -x + s \text{ and } x > 0\}$ and $P_s^- = \{(x,t) \mid t < x + s \text{ and } x < 0\}$ for $s \leq 0$. The dual construction, X^-, produces analogous boundary points $F_s^+ = \{(x,t) \mid t > x + s \text{ and } x > 0\}$ and $F_s^- = \{(x,t) \mid t > -x + s \text{ and } x < 0\}$, for $s < 0$, and a single $F_0 = \{(x,t) \mid t > |x|\}$.
The combining of these in $X^{\#}$ identifies $P_s^+ = F_s^+$ (call it P_s^+) and $P_s^- = F_s^-$

(call it B_s^-), both for $s < 0$, and also $P_0^+ \equiv F_0 \equiv P_0^-$ (call it B_0). Then $I^{-\#}(B_0)$ consists of the points of X making up $P_0^+ \cup P_0^-$ (plus boundary points B_s^+ and B_s^- and others at timelike and null past infinity), two separate components of the past of B_0. Thus, B_0 is a non-regular point in $X^\#$.

The topology to be defined in this Section on a chronological set X works rather well if X is regular. But it can fail to give expected results for non-regular points such as B_0 in $X^\#$ above: It fails to have B_0 as a limit of the sequence $\{(0, 1/n)\}$. A more complicated version of the topology will be detailed in Section 4 to take care of non-regular points.

What is an appropriate topology to place on a chronological set X? One that might come to mind is the Alexandrov topology: Declare $I^-(x)$ and $I^+(x)$ to be open sets for all $x \in X$, as a sub-basis for a topology. For spacetimes, being strongly causal is equivalent to the Alexandrov topology being the same as the manifold topology (see [BE]), which makes this seem a natural choice for the topology on a chronological set. However, this will not serve: Although $I^-(x)$ is always open in any spacetime, there are instances of spacetime-with-boundary in which we do not want that to be so.

A typical example would be any of a number of completions of Minkowski 2-space with a spacelike half-line deleted, say, $X = \mathbb{L}^2 - \{(x, 0) | x \leq 0\}$ (see Figure 2); as completion $\bar{X}$ take X plus points on the future and past edges of the slit: $\{p_s^+ \mid s < 0\}$ on the future side (essentially, $p_s^+ = (s, 0^+)$), $\{p_s^- \mid s < 0\}$ on the past edge ($p_s^- = (s, 0^-)$), and p_0 joining the two (functioning as $(0, 0)$). The topology of the added points is that of two half-lines conjoined at the ends, glued to the $\{t > 0\}$ and $\{t < 0\}$ regions in the obvious manner. As chronology relation on $\bar{X}$, set $p \ll q$ iff there is a future-directed timelike curve from p to q. Then the $\bar{X}$-past of $(0, 1)$ contains $\{p_s^+ \mid -1 < s < 0\}$ and p_0, but not a single p_s^-: $p_0 \in I^-_{\bar{X}}((0, 1))$ but no neighborhood of p_0 is in that past set.

So while the Alexandrov topology is sometimes reasonable (for instance, in some spacetimes-with-boundary with only timelike boundaries), we must do something else for a general chronological set.

The procedure followed here will be to define a topology on a chronological set X by defining what the limits of sequences are; then a subset of X is defined to be closed if and only if it contains the limits of all its sequences. This is not quite as straight-forward as it sounds, if it turns out that a sequence can have more than one limit—and, unfortunately, that is a very live possibility, even for a chronological set which is, say, the future completion of a reasonable spacetime (an example will be given of such). In such a case, one can conceivably have a point which is in the closure of a sequence but is not a "limit" of that sequence as given in the definition.

Perhaps the best way to be clear about what is going on is to eschew the word "limit" and speak simply of a function $L : \mathcal{S}(X) \to \mathfrak{P}(X)$, where $\mathcal{S}(X)$ denotes the set of sequences in X (i.e., maps $\sigma : \mathbb{Z}^+ \to X$) and $\mathfrak{P}(X)$ is the power set of X; $L(\sigma)$ is to be thought of as the set of points which are "first-order" limits of the sequence σ (we can call L the "limit-operator" for X, without prejudice to the term "limit"). So long as L has the property that for subsequences $\tau \subset \sigma$, $L(\tau) \supset L(\sigma)$, then a topology is defined on X by defining a subset A of X to be closed if and only if for every sequence $\sigma \subset A$, $L(\sigma) \subset A$. Every second-countable topological space can have its topology characterized in this fashion—and it is the existence of the countable subset S in the definition of chronological set given in Section 1

that makes it possible to treat chronological sets as second-countable topological spaces.

A singleton set $\{x\}$ will be closed so long as $L(\hat{x})$ is either $\{x\}$ or $\emptyset$, where $\hat{x}$ is the constant sequence $\hat{x}(n) = x$, all n. Let us assume that for all x, $L(\hat{x})$ contains x. Then with $L[A]$, for $A \subset X$, denoting $\bigcup\{L(\sigma) \mid \sigma \in \mathcal{S}(A)\}$, we have $L[A] \supset A$ for any subset A. So long as for any sequence σ, $L(\sigma)$ is finite, then for any set A, *closure*$(A) = L[A]$. More generally, we must look to iterations of the set-function $L[\ ]$: $L^1 = L$; for any ordinal α let $L^{\alpha+1}[A] = L[\,L^\alpha[A]\,]$; and for any limit-ordinal α, let $L^\alpha[A] = \bigcup_{\beta<\alpha} L^\beta[A]$. Then *closure*$(A) = L^\Omega[A]$, where Ω is the first uncountable ordinal (reason: Any point $x \in L^{\Omega+1}[A]$ lies in $L(\sigma)$ for some sequence σ lying in $L^\Omega[A]$. For all n, there is some ordinal $\alpha_n < \Omega$ with $\sigma(n) \in L^{\alpha_n}[A]$. But there is some $\beta < \Omega$ with all $\alpha_n < \beta$. Thus, $x \in L^\beta[A] \subset L^\Omega[A]$. Therefore, $L^{\Omega+1}[A] = L^\Omega[A]$.). For a function $f : X \to Y$, where X and Y are topological spaces defined using, respectively, the limit-operators L_X and L_Y, f is continuous if and only if for every sequence σ in X, $f[L_X(\sigma)] \subset L_Y^\Omega[\,f[\sigma]\,]$. (This follows thus: x is a limit of a sequence σ if and only if every open set containing x eventually contains σ, i.e, every closed set excluding x eventually excludes σ, i.e., every closed set containing a subsequence of σ contains x; this is equivalent to x being in the closure of every set containing a subsequence of σ, which is equivalent to x being in the closure of every subsequence of σ, i.e., $x \in \bigcap_{\tau\subset\sigma} L^\Omega[\tau]$. Then $f : X \to Y$ is continuous if and only if for every sequence σ and point x in X, $x \in \bigcap_{\tau\subset\sigma} L_X^\Omega[\tau]$ implies $f(x) \in \bigcap_{\tau\subset f[\sigma]} L_Y^\Omega[\tau] = \bigcap_{\tau\subset\sigma} L_Y^\Omega[f\,[\tau]\,]$; and this is equivalent to having, for every sequence σ in X, $f[\,L_X^\Omega[\sigma]\,] \subset L_Y^\Omega[f\,[\sigma]\,]$. A bit of transfinite induction shows that if for every sequence σ, $f[L_X(\sigma)] \subset L_Y^\Omega[\,f[\sigma]\,]$, then for every sequence σ, $f[\,L_X^\Omega[\sigma]\,] \subset L_Y^\Omega[\,f[\sigma]\,]$.) In particular, $f[L_X(\sigma)] \subset L_Y(f[\sigma])$ for all σ implies f is continuous.

Limit-operator in a regular chronological set. For X a regular chronological set, the limit-operator L is defined thus:

For any sequence $\sigma = \{x_n\}$ and any point x, $x \in L(\sigma)$ if and only if

(1) for all $y \ll x$, eventually $y \ll x_n$ (i.e., for some n_0, $y \ll x_n$ for all $n > n_0$), and
(2) for any IP P containing $I^-(x)$, if for all $y \in P$, there is some subsequence $\{x_{n_k}\}$ with $y \ll x_{n_k}$ for all k, then $P = I^-(x)$.

Note, first of all, that this obeys $L(\tau) \supset L(\sigma)$ for $\tau \subset \sigma$: Clause (2) says, essentially, that $I^-(x)$ is a maximal IP for obeying clause (1)—i.e., that all of its elements are eventually in the past of the sequence—but this is generalized to subsequences. (Sometimes the contrapositive of clause (2) is more natural: Any IP properly containing $I^-(x)$ contains a point y such that eventually $y \not\ll x_n$.) It follows that L defines a topology on X; as this topology is designed especially with $\hat{X}$ in mind, let us call it the $\widehat{\ }$-*topology* on X. We will call the elements of $L(\sigma)$ the $\widehat{\ }$-*limits* of σ. (See Figure 3.)

Next, note that $L(\sigma)$ depends only on the points which make up the sequence σ, and not on the order in which they appear in the sequence (save for whether a given point appears a finite number of times or an infinite number of times in the sequence): We could reformulate the definition as $x \in L(\sigma)$ if and only if (1) for all $y \ll x$, y is in the past of all but a finite number of points of σ (more precisely: $\{x_n \mid y \not\ll x_n\}$ is finite), and (2) for any IP $P \supset I^-(x)$, if for all $y \in P$, y is in the

past of an infinite number of points of σ (more precisely: $\{n \mid y \ll x_n\}$ is infinite), then $P = I^-(x)$.

2.1 Proposition. *For any regular past-distinguishing chronological set X, for any $x \in X$, $L(\hat{x}) = \{x\}$; thus, all points are closed in the $\hat{}$-topology.*

Proof. First, $x \in L(\hat{x})$: For any $y \ll x$, $y \ll \hat{x}(n)$ for all n (i.e, $y \ll x$). Let P be any IP with $I^-(x) \subset P$. To say that for any $y \in P$, y is in the past of each element of a subsequence of $\hat{x}$ is to say $y \ll x$ for all $y \in P$, i.e., that $P \subset I^-(x)$; thus, this implies $P = I^-(x)$.

Second, if $z \in L(\hat{x})$, then $z = x$: We have for all $y \ll z$, $y \ll x$, i.e., $I^-(z) \subset I^-(x)$. Thus we can apply part (2) of the definition of $z \in L(\hat{x})$ to the IP $I^-(x)$: For all $y \in I^-(x)$, y is in the past of each element of a subsequence of $\hat{x}$, so $I^-(x) = I^-(z)$. Thus, by past-distinguishment, $z = x$. $\square$

An immediate corollary is that in a past-distinguishing regular chronological set, if a sequence σ has more than one point that appears an infinite number of times, then $L(\sigma) = \emptyset$ (if both x and y appear infinitely often, then $\hat{x}$ and $\hat{y}$ are both subsequences, so $L(\sigma) \subset \{x\}$ and $L(\sigma) \subset \{y\}$).

Here is an example showing that a most reasonable chronological set need not be Hausdorff (see figure 4): Let M be the the spacetime considered before, $\mathbb{L}^2$ with the negative time-axis removed, and let $X = \hat{M}$, its future completion; this introduces the boundary points mentioned before, P_s^+ and P_s^-, for $s \leq 0$, representable respectively by $(s, 0^+)$ and $(s, 0^-)$; however, we will not meld P_0^+ and P_0^- together, but keep them separate, staying precisely with $\hat{M}$. Consider the sequence $\sigma = \{(0, 1/n)\}$: The past of P_0^+ (in X) consists of $\{(x, t) \mid 0 < x < -t\} \cup \{P_s^+ \mid s < 0\}$; for all $p \in I_X^-(P_0^+)$, for all n, $p \ll \sigma(n)$. Furthermore, for any IP P (in X) properly containing $I_X^-(P_0^+)$, eventually $\sigma(n) \in P$, and we can find $p \in P$ so that for n sufficiently high, $p \ll \sigma(n)$ fails. Therefore $P_0^+ \in L(\sigma)$. It follows that any closed set not containing P_0^+ must also omit all but a finite portion of σ: Any neighborhood of P_0^+ must contain a tail-end of σ. The same, of course, is true for P_0^-, so any two neighborhoods of these points intersect. (Actually, any neighborhood of either point must contain an $\mathbb{L}^2$-neighborhood of the origin intersected with $\{(x, t) \mid |x| < t\}$.)

We want to know that the concept of $\hat{}$-limit is compatible with future limit of a future chain. This is the case:

2.2 Proposition. *Let $c = \{x_n\}$ be a future chain in a regular chronological space. Then a point x is a future limit for c if and only if it is a $\hat{}$-limit for c; furthermore, if for every point x, $L(\hat{x}) = \{x\}$, then $L^\Omega[c] = L[c]$.*

Proof. Suppose that x is a future limit for c, i.e., that $I^-(x) = I^-[c]$; we need to show that it is also a $\hat{}$-limit. For any $y \ll x$, $y \in I^-[c]$, so $y \ll x_n$ for all n large enough. Consider any IP $P \supset I^-(x)$: For any point y, if, for all k, $y \ll x_{n_k}$ for some subsequence, then $y \in I^-[c] = I^-(x)$; thus, if this is true for all $y \in P$, it follows that $P = I^-(x)$.

Now suppose that $x \in L(c)$; we must show it a future limit of c. For all $y \ll x$, we have eventually $y \ll x_n$; thus $I^-(x) \subset I^-[c]$. Now, for all $y \in I^-[c]$, eventually $y \ll x_n$, and $I^-[c]$ is an IP containing $I^-(x)$; therefore, applying clause (2) of the definition of $L(c)$, we obtain $I^-[c] = I^-(x)$.

To show $L^\Omega[c] = L[c]$, we need only show that $L^2[c] = L[c]$ (which is $c \cup L(c)$). Consider any sequence of points $\sigma = \{x_n\}$ in $L(c)$. We just need to show that for any $x \in L(\sigma)$, x is a future limit of c, for then (from what we've just seen) $x \in L(c)$. Consider such $x \in L(\sigma)$: For any $y \ll x$, eventually $y \ll x_n$; since $I^-(x_n) = I^-[c]$, that shows $y \in I^-[c]$. Therefore, $I^-(x) \subset I^-[c]$. Thus, $I^-[c]$ is an IP containing $I^-(x)$. If it properly contains $I^-(x)$, then it must contain some y which is eventually not in the past of x_n; but that is impossible, since x_n and c have the same past, for any n. Therefore, $I^-[c] = I^-(x)$, i.e., x is a future limit of c. $\square$

It follows that a chronological function $f : X \to Y$ between regular chronological sets with Y past-distinguishing, which is continuous in the respective $\widehat{\ }$-topologies, is also future-continuous: For any future chain c in X with future limit x, x is also a $\widehat{\ }$-limit for c; therefore, $f(x) \in L[\,f[c]\,]$. Now, $f[c]$ is also a future chain in Y, so $L(f[c])$ consists of the future limits of $f[c]$; with Y past-distinguishing, there can be no more than one of these: There is at most a single $\widehat{\ }$-limit for $f[c]$. Thus, $f(x)$ is that unique $\widehat{\ }$-limit for $f[c]$, which must be the (unique) future limit of $f[c]$. However, the converse does not hold: A future-continuous function can fail to be $\widehat{\ }$-continuous. A simple example is $f : \mathbb{L}^1 \to \mathbb{L}^1$ with $f(t) = t$ for $t \leq 0$ and $f(t) = t + 1$ for $t > 1$.

There are many ways to define a topology for chronological sets; why should this $\widehat{\ }$-topology be considered a good choice, especially since it need not even be Hausdorff in reasonable instances? One of the prime prerequisites of a good topology construction is that it replicate the manifold topology in the case of a spacetime; that is obeyed by the $\widehat{\ }$-topology:

2.3 Theorem. *Let M be a strongly causal spacetime; then the $\widehat{\ }$-topology induced by the spacetime chronology relation is the same as the manifold topology on M.*

Proof. It suffices to show that for any sequence $\sigma = \{x_n\}$ in M, a point x is the usual (manifold-)topological limit of σ if and only if it is a $\widehat{\ }$-limit of σ (for that establishes that the map $1_M : (M, \text{manifold topology}) \to (M, \widehat{\ }\text{-topology})$ is bicontinuous).

Suppose x is the usual topological limit of σ. Clause (1): For any $y \ll x$, $V = I^+(y)$ is an open set containing x; therefore, the sequence is eventually inside V, i.e., eventually $y \ll x_n$. Clause (2) (see Figure 5): Let P be any IP containing $I^-(x)$. Since M is strongly causal, it has the Alexandrov topology, and we can pick points $z \ll x$ (note that $z \in P$) and $w \gg x$ so that $U = I^+(z) \cap I^-(w)$ is geodesically convex: For any $u \ll v$ both in U, there is a future-timelike geodesic in U from u to v. Suppose that every point in P is in the past of every element of a subsequence of σ. For any $p \in P$, there is some $q \in P$ with $q \gg p$ and $q \gg z$. For some subsequence, for all k, $q \ll x_{n_k}$; also, $x_{n_k} \in U$. Therefore, for all k, q and x_{n_k} are both in U, so there is a future-timelike geodesic γ_k from q to x_{n_k}, lying in U. Then the curves $\{\gamma_k\}$ have a limit curve γ (lying in U) from q to x, and γ is a future-causal geodesic. Then we have $p \ll q \prec x$, so $p \ll x$: $P = I^-(x)$.

Now suppose x is a $\widehat{\ }$-limit of σ (see Figure 6). If x is not the manifold-topological limit of σ, then there are a relatively compact neighborhood U of x and a subsequence $\{x_{n_k}\}$ which never enters U; by strong causality we can pick U so that no timelike curve exits and re-enters U. Pick a future chain $\{z_n\}$ with x as future limit (which is the same as saying the manifold-topological limit). For n sufficiently

large, $z_n \in U$; then, since $z_n \ll x$, for k sufficiently large—say, $k \geq K_n$—$z_n \ll x_{n_k}$, so there is a future-timelike curve c^n_k from z_n to x_{n_k}. Since x_{n_k} is not in U, c^n_k exits U at a point $y^n_k \in \partial U$. For each n, the curves $\{c^n_k \mid k \geq K_n\}$ have a future-causal limit curve c^n from z_n to some point $y^n \in \partial U$. The curves $\{c^n\}$ have a future-causal limit curve c from x to some point $y \in \partial U$.

Let $P = I^-[c] = I^-(y)$ and apply clause (2) of the $\widehat{\ }$-limit definition (applicable because any point in the past of x is in the past of y): For any $p \in P$, we have $I^+(p)$ is a neighborhood of y; therefore, for n sufficiently large $y^n \in I^+(p)$; therefore, for k sufficiently large—say, $k \geq J_n$—$y^n_k \in I^+(p)$. Since $y^n_k \ll x_{n_k}$ (via c^n_k), we have that for n sufficiently large and $k \geq J_n$, $p \ll x_{n_k}$: Clause (2) allows us to conclude $P = I^-(x)$. But this is impossible, since $y \succ x$ and $y \neq x$, since $y \in \partial U$ with U a neighborhood of x. Therefore, x must be the manifold-limit of σ. $\square$

Another desirable property of a topology construction is that it accord well with boundary constructions: If $\bar{X}$ is X with some sort of future boundary, then $\bar{X}$ should have a topology appropriately related to that of X. Here is how the $\widehat{\ }$-topology fares in this regard:

2.4 Theorem. *Let $\bar{X}$ be a regular chronological set with X a subset of $\bar{X}$ satisfying the following:*

1. *The restriction of $\ll$ to X yields another regular chronological set; and*
2. *for any $p \ll q$ in $\bar{X}$, there is some $x \in X$ so that $p \ll x \ll q$.*

Then the $\widehat{\ }$-topology on X (as a chronological set in its own right) is the same as the subspace topology it inherits from the $\widehat{\ }$-topology on $\bar{X}$, and X is dense in $\bar{X}$.

Proof. We must first establish a correspondence between the IPs of X and those of $\bar{X}$:

For any IP P in X, generated by a future chain c in X, let $\bar{P} = I^-_{\bar{X}}[c]$, an IP in $\bar{X}$. Since $\bar{P} = I^-_{\bar{X}}[P]$, this is independent of the choice of the generating chain c.

For any IP Q in $\bar{X}$, generated by a future chain c in $\bar{X}$, use condition (2) of the hypotheses to construct an interweaving chain c_0 in X (i.e., if $c = \{p_n\}$, then pick $c_0 = \{x_n\}$ with $p_n \ll x_n \ll p_{n+1}$); let $Q_0 = I^-_X[c_0]$, an IP in X. Since $Q_0 = Q \cap X$, this is independent of the choices of the chains c and c_0.

Lemma. *The maps $P \mapsto \bar{P}$ and $Q \mapsto Q_0$ establish an isomorphism between the IPs of X and of $\bar{X}$, as partially ordered sets under inclusion.*

Proof of Lemma. Let P be an IP in X, generated by a future chain c. Then $\bar{P}$ is generated (in $\bar{X}$) by the same chain c. Thus, for a generating chain for $(\bar{P})_0$, we may take yet again the chain c (or, following strictly the defining construction, a chain c_0 in X interweaving c—having, therefore, the same past as c). Thus, $(\bar{P})_0 = P$.

Let Q be an IP in $\bar{X}$, generated by a future chain c, with an interweaving chain c_0 in X. Then $\overline{Q_0}$ has c_0 as generating chain in $\bar{X}$. Since c_0 and c are interweaving, they have the same past in $\bar{X}$, i.e., $\overline{Q_0} = Q$.

If $P \subset P'$ are IPs in X, then $\bar{P} = I^-_{\bar{X}}[P] \subset I^-_{\bar{X}}[P'] = \overline{P'}$.

If $Q \subset Q'$ are IPs in $\bar{X}$, then $Q_0 = Q \cap X \subset Q' \cap X = Q'_0$. $\square$

Consider a sequence $\sigma = \{x_n\}$ in X. We will see that for any $x \in X$, $x \in L_X(\sigma)$ if and only if $x \in L_{\bar{X}}(\sigma)$, i.e., $L_X(\sigma) = L_{\bar{X}}(\sigma) \cap X$. With $i : X \to \bar{X}$ denoting the inclusion map, with the respective $\widehat{\ }$-topologies, that will establish that i is a

homeomorphism onto its image, i.e., that the $\hat{}$-topology on X is the same as its subspace topology in $\bar{X}$.

Suppose x is a $\hat{}$-limit of σ with respect to X, i.e., (1) for all $y \in X$ with $y \ll x$, eventually $y \ll x_n$, and (2) for any IP P in X with $P \supset I^-_X(x)$, if for all $y \in P$, for all elements of some subsequence, $y \ll x_{n_k}$, then $P = I^-_X(x)$. We want to establish the same results for $\bar{X}$.

Clause (1): For any $p \in \bar{X}$ with $p \ll x$, there is some $y \in X$ with $p \ll y \ll x$ (hypothesis 2). We know eventually $y \ll x_n$, whence follows $p \ll x_n$.

Clause (2): Let Q be any IP in $\bar{X}$ containing $I^-_{\bar{X}}(x)$. Then $Q_0 = Q \cap X$ is an IP in X containing $I^-_{\bar{X}}(x) \cap X = I^-_X(x)$. Suppose for all $q \in Q$, for all k, $q \ll x_{n_k}$; then the same is true for Q_0, whence $Q_0 = I^-(x)$. Then $Q = \overline{Q_0} = \overline{I^-_X(x)} = I^-_{\bar{X}}(x)$ (the last can be seen by noting that $(I^-_{\bar{X}}(x))_0 = I^-_{\bar{X}}(x) \cap X = I^-_X(x)$). That finishes showing $x \in L_{\bar{X}}(\sigma)$.

Now suppose $x \in X$ is a $\hat{}$-limit of σ with respect to $\bar{X}$, i.e., (1) for all $q \ll x$, eventually $q \ll x_n$, and (2) for any IP Q in $\bar{X}$, if for all $q \in Q$, for all k, $q \ll x_{n_k}$, then $Q = I^-_{\bar{X}}(x)$. We need to see the same holds with respect to X:

Clause (1): For any $y \in X$ with $y \ll x$, we clearly have eventually $y \ll x_n$. Clause (2): Let P be any IP in X with $P \supset I^-_X(x)$. Then $\bar{P}$ is an IP in $\bar{X}$ and $\bar{P} \supset \overline{I^-_X(x)} = I^-_{\bar{X}}(x)$. Suppose for all $z \in P$, for some subsequence, $z \ll x_{n_k}$. For any $q \in \bar{P}$, there is some $p \in P$ with $q \ll p$; then there is some $y \in X$ with $q \ll y \ll p$. We have $y \in P$, since $y \ll p$ places y in $\bar{P}$, and $y \in X$. Therefore, for all k, $y \ll x_{n_k}$, whence $q \ll x_{n_k}$ also. It follows that $\bar{P} = I^-_{\bar{X}}(x)$, so $P = (\bar{P})_0 = (I^-_{\bar{X}}(x))_0 = I^-_X(x)$. Thus is x shown to be in $L_X(\sigma)$.

Finally, consider any $p \in \bar{X}$. By regularity, $I^-_{\bar{X}}(p)$ is an IP in $\bar{X}$, generated by a future chain c, so $(I^-_{\bar{X}}(p))_0$ is an IP in X, generated by an interweaving future chain c_0 in X (as in the proof of the Lemma). Note that p is a future limit of c_0 (in $\bar{X}$), since interweaving future chains have the same past. Then, by Proposition 2.2, p is a $\hat{}$-limit of c_0 (in $\bar{X}$). It follows that p is in the closure of X. □

We can apply this to the future completion of a regular chronological set:

2.5 Corollary. *Let X be a regular chronological set. Then the standard future injection $\hat{\imath}_X : X \to \hat{X}$ is a homeomorphism onto its image, and X is dense in $\hat{X}$.*

Proof. By the definition of $\ll$ in $\hat{X}$, its restriction to the elements of X is precisely the original chronology relation on X. Since X is regular, so is $\hat{X}$. Again using the definition of $\ll$ in $\hat{X}$, we see that for any $P \ll Q$ in $\hat{X}$, there is some $x \in X$ with $P \ll x \ll Q$: For instance, if P and Q are in $\hat{\partial}X$, then there is some $y \in Q$ with $P \subset I^-(y)$; then there is some $x \in Q$ with $y \ll x$, and we have $P \ll x \ll Q$. Thus, Theorem 2.4 applies. □

If $\hat{\partial}X$ is to have the expected properties of a boundary for X, then not only should it be in the closure of X, but it should itself be closed. Now, this cannot be true in general: X could, for instance, consist of a spacetime M plus a portion of $\hat{\partial}(M)$, in which case $\hat{\partial}(X)$ is the remainder of $\hat{\partial}(M)$ and need not be closed. (Simple example: $M = \mathbb{L}^2 - \{(x,0) \mid x \leq 0\}$, so $\hat{\partial}(M)$ consists of the elements of $\hat{\partial}(\mathbb{L}^2)$ plus $\{(x,0) \mid x \leq 0\}$ (as elements attached to the "underside" of the slit). If X is M with $\{(0,0)\}$ adjoined, then $\hat{\partial}(X) = \{(x,0) \mid x < 0\}$ and has $(0,0)$ in its closure, though $(0,0)$ is not in $\hat{\partial}(X)$.)

But so long as we restrict ourselves to spacetimes—with no boundary elements conjoined—then the Future Chronological Boundary is, indeed, closed:

2.6 Proposition. *Let M be a strongly causal spacetime. Then $\hat{\partial}(M)$ is closed in $\hat{M}$.*

Proof. Let $\sigma = \{P_n\}$ be a sequence of elements of $\hat{\partial}(M)$; we must show that it is impossible for an element $x \in M$ to be in $L(\sigma)$. (It then follows that $L^{\Omega}[\hat{\partial}(M)] \subset \hat{\partial}(M)$.)

For suppose $x \in L(\sigma)$. By strong causality we have a relatively compact neighborhood U of x, such that no causal curve exits and re-enters U. Each P_n is the past of a future-endless curve τ_n; each τ_n is eventually outside U. For all $z \ll x$, eventually $z \ll P_n$ (clause (1) of the definition of L), i.e., $z \in P_n$; thus, there is a future-timelike curve from z to a point on τ_n, which we can choose to be on a portion of τ_n which is not in U and which from that point on never enters U.

We are now in exactly the same position as in the latter part of the proof of Theorem 2.3: With choice of a future chain $\{z_n\}$ approaching x, applying the paragraph above to the points $\{z_n\}$ yields future-timelike curves c^n_k from z_n to τ_k outside of U, intersecting ∂U in y^n_k; these curves have a subsequence converging to a future-causal c from x to a point $y \in \partial U$. Every point in $I^-(y)$ is in the past of that subsequence, so, by clause (2) of the definition of L, $I^-(y) = I^-(x)$, implying $y = x$, which is false. $\square$

Section 5 will give the example of a large class of spacetimes M with spacelike boundary (including interior Schwarzschild) in which the $\widehat{\ }$-topology gives an entirely reasonable topology for $\hat{M}$ (the intuitively "right" topology). It will also be shown that for spacetimes M with spacelike boundary, the $\widehat{\ }$-topology agrees with the topology given to a nice completion $\bar{M}$ formed by embedding M into a larger manifold.

But the true value of the $\widehat{\ }$-topology construction is that it leads to a categorical result, an extension into a topological category of the categorical results for the Future Chronological Boundary as a construction among chronological sets. The basic strategy is to define a new set of categories, subcategories of **Chron** and its allied categories, by restricting the morphisms to $\widehat{\ }$-continuous functions. For now, we will look at (past-)regular chronological sets: Let **PregFtopChron** be the category whose objects are (past-)regular chronological sets and whose morphisms are $\widehat{\ }$-continuous, future-continuous functions (**Ftop** denoting "topology defined with future boundaries in mind"). Our task is to show that all manner of future-continuous functions, needed for the categorical results of [H], are $\widehat{\ }$-continuous, also. Then the same form of categorical results will apply in the new, topological, categories; Corollary 2.5 was the first of such results.

However, there is a substantial fly in the ointment: It is not true, in general, that the extension of a $\widehat{\ }$-continuous function to future completions remains $\widehat{\ }$-continuous—one can have $f : X \to Y$ future-continuous and $\widehat{\ }$-continuous, but $\hat{f} : \hat{X} \to \hat{Y}$, though future-continuous, fail to be $\widehat{\ }$-continuous. Essentially, this can happen if X has a timelike boundary. Here's an example (see Figure 7):

Let $X = \{(x,t) \in \mathbb{L}^2 \mid x > 0\}$, and let $Y = \mathbb{L}^2$. Define $f : X \to Y$ by

$$f(x,t) = \begin{cases} (x,t), & t \le 0 \\ (x,(1+1/x)t), & 0 \le t \le x \\ (x,t+1), & t \ge x \end{cases} .$$

It is easy to examine $f_*(\frac{\partial}{\partial t})$ and $f_*(\frac{\partial}{\partial x})$ and see thereby that f_* carries the two future-null vectors $\frac{\partial}{\partial t} + \frac{\partial}{\partial x}$ and $\frac{\partial}{\partial t} - \frac{\partial}{\partial x}$ into future-causal vectors, in each of the regions where f is differentiable. Since f is continuous, it follows that it is chronological. The continuity of f in the obvious sense is equivalent to the $\hat{\ }$-continuity of f (Theorem 2.3). Thus, from the observation following Proposition 2.2, it follows that f is future-continuous (since Y is evidently past-distinguishing).

Now consider $\hat{f} : \hat{X} \to \hat{Y}$. The Future Chronological Boundary $\hat{\partial}(X)$ consists of the right half of future null infinity, $\mathfrak{I}^+_R$; future timelike infinity, i^+; and a timelike boundary component that can be conventionally described as $\{(0,t) \mid t \in \mathbb{R}\}$; the $\hat{\ }$-topology is just as is to be expected. We have the usual future boundary for $\mathbb{L}^2$: $\hat{\partial}(Y)$ consists of future null infinity $\mathfrak{I}^+$ and future timelike infinity i^+, with the expected topology. On $\mathfrak{I}^+_R$ and i^+, $\hat{f}$ does the expected, essentially the identity map. But on the other boundary points, we have

$$\hat{f}(0,t) = \begin{cases} (0,t), & t \le 0 \\ (0,t+1), & t > 0 \end{cases} .$$

Thus, $\hat{f}$ is not $\hat{\ }$-continuous.

So, what to do? It appears that what allows things to go awry is $\hat{f}$ carrying boundary elements into points (including boundary points) that are timelike-related. So we will take the following tack in constructing a subcategory where this does not happen: We'll constrain our objects to have only "spacelike boundaries", and we'll insist that $\hat{f}$ preserve the spacelike nature of such.

To this end, let us call a regular point x in a chronological set X *inobservable* (in X) if the only IP which contains $I^-(x)$ is $I^-(x)$ (non-regular points will be considered in Section 4); we will say a chronological set X *has only spacelike boundaries* if (1) all elements of $\hat{\partial}(X)$ are inobservable (in $\hat{X}$) and (2) the inobservables in $\hat{X}$ form a closed subset, or $\hat{\partial}(X)$ is closed in $\hat{X}$ (the reason for (2) is technical in nature, allowing proofs to go through); and a chronological map $f : X \to Y$ will be said to *preserve spacelike boundaries* if $\hat{f}$ (not just f) preserves inobservables (i.e., for any p inobservable in $\hat{X}$, whether boundary element or otherwise, $\hat{f}(p)$ is inobservable in $\hat{Y}$). Then we define the category **PregSpbdFtopChron** to have as objects regular chronological sets with only spacelike boundaries, and to have as morphisms future-continuous, $\hat{\ }$-continuous maps which preserve spacelike boundaries.

What are the objects of **PregSpbdFtopChron**? Suppose M is a spacetime and X is M with some elements of $\hat{\partial}(M)$ conjoined; then $\hat{\partial}(X)$ consists of the remaining elements of $\hat{\partial}(M)$, i.e., those not already in X; $\hat{X}$ is the same as $\hat{M}$, and the inobservables there are all in $\hat{\partial}(M)$ (since any point of a spacetime is observable). Thus, to say X has only spacelike boundaries is to say (1) the remaining elements of $\hat{\partial}(M)$—those not already in X—are all inobservable, and (2) either those remaining elements of $\hat{\partial}(M)$ form a closed subset, or the entire collection of inobservables in $\hat{M}$ (i.e., in $\hat{\partial}(M)$) forms a closed set. Thus, for instance, M might have a timelike

portion in its boundary, so long as X includes the timelike components of $\hat{\partial}(M)$; but either all the inobservable boundary points are required to form a closed set, or the non-included boundary points must do so. In the case that M has only inobservable boundary points to begin with, it doesn't matter which boundary points, if any, are included in X, as then the set of inobservables is the same as $\hat{\partial}(M)$, and that is always closed (Proposition 2.6). In any case, for any chronological set X, $\hat{X}$ has only spacelike boundaries, as $\hat{\partial}(\hat{X})$ is empty (and, hence, closed).

It is fruitful to note that a strongly causal spacetime M which has only spacelike boundaries is necessarily past-determined: For $I^-(x) \subset I^-(w)$, let $\{x_n\}$ be a future chain with x as future limit; then each $x_n \ll w$, so there is a future-directed timelike curve γ_n from x_n to w. The curves $\{\gamma_n\}$ have a limit curve γ, which is a future-directed causal curve beginning at x; either γ terminates at w or it has no future endpoint. In the latter case, γ generates an IP P which is in $\hat{M}$; but then P is a proper subset of $I^-(w)$, so P is observable. So if M has only spacelike boundaries, γ must reach w, whence $x \prec w$. Thus, if $w \ll y$, we have $x \prec w \ll y$, so $x \ll y$.

But for chronological sets in general, having only spacelike boundaries does not imply past-determination: If M is not past-determined, then neither is $\hat{M}$, but the latter has only spacelike boundaries.

What are the morphisms of **PregSpbdFtopChron**? For $f : X \to Y$ to be in this category, it is required that $\hat{f}$ preserve inobservables: This is a restriction not just on f, but also on what $\hat{f}$ does to elements of $\hat{\partial}(X)$. Thus, for instance, if X is lower-$\mathbb{L}^2$, $\{(x,t) \mid t < 0\}$, we cannot have Y as $\mathbb{L}^2$ and f as the inclusion map, as $\hat{f}$ carries inobservables of $\hat{X}$ into observables in $\hat{Y}$. The effect of this restriction is to require, in essence, that we consider only maps which take boundaries to boundaries, not to interiors.

We need to note that **PregSpbdFtopChron** is, indeed, a category: For any X, $\hat{1}_X = 1_{\hat{X}}$, so $\hat{1}_X$ preserves inobservables. For $f : X \to Y$ and $g : Y \to Z$, if $\hat{f}$ and $\hat{g}$ both preserve inobservables, so does $\widehat{g \circ f} = \hat{g} \circ \hat{f}$.

Next we need to see that future completion is a functor for the appropriate subcategory; in particular, for $f : X \to Y$ in our **SpbdFtop** category, we need $\hat{f} : \hat{X} \to \hat{Y}$ also $\hat{\ }$-continuous. From Proposition 6 in [H], one would expect that we need to have Y past-determined in order to have $\hat{f}$ chronological; but, as it turns out, the assumption of spacelike boundaries makes that an unnecessary hypothesis.

Proposition 2.7. *Let $f : X \to Y$ be a map in the category* **PdisPregSpbdFtop-Chron**. *Then $\hat{f} : \hat{X} \to \hat{Y}$ is future-continuous and $\hat{\ }$-continuous.*

Proof. First we consider why $\hat{f}$ is even chronological: This is established in Proposition 6 of [H], under the additional hypothesis that Y is past-determined. That hypothesis is used only to show that for $P \in \hat{\partial}(X)$ and $q \in \hat{X}$ (whether in X or in $\hat{\partial}(X)$), if $P \ll q$, then $\hat{f}(P) \ll \hat{f}(q)$. But since X has only spacelike boundaries, for any $P \in \hat{\partial}(X)$, there is no $q \in \hat{X}$ with $P \ll q$ (that would imply $P \subset \hat{I}^-(q)$, whence $P = \hat{I}^-(q)$, so $P \ll q$ is false). That is the only portion of the proof that uses past-determination. Consequently, $\hat{f}$ is future-continuous.

To show $\hat{\ }$-continuity of $\hat{f}$, consider first a sequence of points $\sigma = \{x_n\}$ in X with $x \in X$ in $L_{\hat{X}}(\sigma)$. From the proof of Theorem 2.4 we know that $L_X(\sigma) = L_{\hat{X}}(\sigma) \cap X$ for σ in X, so this is equivalent to x being in $L_X(\sigma)$. Since f is $\hat{\ }$-continuous, it follows that $f(x) \in L^{\Omega}[f[\sigma]]$

From knowing that, in any regular chronological set Z, $L_Z(\sigma) = L_{\hat{Z}}(\sigma) \cap Z$ for σ in Z, one easily derives that for any subset $A \subset Z$, $L_Z[A] = L_{\hat{Z}}[A] \cap Z$; then an easy transfinite induction establishes that for all ordinals α, $L_Z^\alpha[A] \subset L_{\hat{Z}}^\alpha[A] \cap Z$ (equality fails at $\alpha = 2$: one has $L_Z^2[A] = L_{\hat{Z}}[L_{\hat{Z}}[A] \cap Z] \cap Z$; in addition to $L_Z^2[A]$, $L_{\hat{Z}}^2[A] \cap Z$ also contains $L_{\hat{Z}}[L_{\hat{Z}}[A] \cap \hat{\partial}(Z)] \cap Z$). Thus, in particular, $L_Y^\Omega[\,f[\sigma]\,] \subset L_{\hat{Y}}^\Omega[\,f[\sigma]\,] \cap Y$. Therefore, we have $f(x) \in L_{\hat{Y}}^\Omega[\,f[\sigma]\,]$.

Now consider the same sequence σ with $P \in \hat{\partial}(X)$ in $L_{\hat{X}}(\sigma)$. P is generated by some future chain $c = \{z_m\}$ in X; the IP $Q = I^-[\,f[c]\,]$ in Y is either in $\hat{\partial}(Y)$—in which case $\hat{f}(P) = Q$—or is $I^-(y)$ for some unique $y \in Y$—in which case $\hat{f}(P) = y$. In either case, we have each z_m is eventually in the past of x_n, so the same is true for $f(z_m)$ and $f(x_n)$; this gives us that for any $q \ll \hat{f}(P)$, eventually $q \ll f(x_n)$ (irrespective of whether $q \in Y$ or $q \in \hat{\partial}(Y)$). Now consider any IP Q' in $\hat{Y}$ containing $\hat{I}^-(\hat{f}(P))$ ($\hat{I}^-$ denoting past in $\hat{Y}$): Since P is in $\hat{\partial}(X)$ and X has only spacelike boundaries, P is inobservable; since f preserves spacelike boundaries, $\hat{f}(P)$ is also inobservable; therefore, $Q' = \hat{I}^-(\hat{f}(P))$. This finishes showing that $\hat{f}(P) \in L_{\hat{Y}}(f[\sigma])$.

Now consider a sequence $\sigma = \{P_n\}$ of points in $\hat{\partial}(X)$. Since X has only spacelike boundaries, all the P_n are inobservable. Either the inobservables in $\hat{X}$ form a closed subset, or $\hat{\partial}(X)$ is closed. Thus, for $p \in L_{\hat{X}}(\sigma)$, either p is inobservable (because the P_n are) or p is in $\hat{\partial}(X)$ (because the P_n are); but in the latter case, again p is inobservable, since X has only spacelike boundaries. Since f preserves spacelike boundaries, $\hat{f}(p)$ is also inobservable (irrespective of whether p is in X or in $\hat{\partial}(X)$). Now proceed much as in the last paragraph: Since X is regular, there is some future chain $c = \{z_m\}$ with p as future limit of c (we need the assumption of regularity only in case p might be in X, not in $\hat{\partial}(X)$). Each z_m is eventually in the past of P_n, so $f(z_m)$ is eventually in the past of $\hat{f}(P_n)$ (since f chronological implies that $\hat{f}$ is chronological). As above, this gives us that for any $q \ll \hat{f}(p)$, eventually $q \ll \hat{f}(P_n)$. Since p is inobservable, as before, no IP can properly contain $\hat{I}^-(\hat{f}(p))$. Therefore, $\hat{f}(p) \in L_{\hat{Y}}(\hat{f}[\sigma])$.

Finally consider a general sequence $\sigma = \{p_n\}$ in $\hat{X}$ with $p \in L_{\hat{X}}(\sigma)$. If σ contains a subsequence σ' lying in $\hat{\partial}(X)$, then since p is also in $L_{\hat{X}}(\sigma')$, it must be that p is inobservable (as explained in the previous paragraph). As before, for all $q \ll \hat{f}(p)$, eventually $q \ll \hat{f}(p_n)$; there is no IP properly containing $\hat{I}^-(\hat{f}(p))$; and, therefore, $\hat{f}(p) \in L_{\hat{Y}}(\hat{f}[\sigma])$. If, on the other hand, there is no subsequence of σ lying in $\hat{\partial}(X)$, then, except for a finite number of points, which can be ignored, σ lies in X, and the first two cases apply. $\square$

2.8 Corollary. *Future completion* $\hat{}$ *is a functor from* **PdisPregSpbdFtopChron** *to* **FcplPdisPregSpbdFtopChron**.

Proof. We need to see that if X is past-distinguishing, regular, and has only spacelike boundaries, then the same is true for $\hat{X}$. The only issue is spacelike boundaries; but this is trivial, since $\hat{\partial}(\hat{X})$ is empty (and, hence, is closed). We also need to see that if f is future-continuous and $\hat{}$-continuous and preserves spacelike boundaries, then the same is true for $\hat{f}$. By Proposition 2.7, the only issue is preserving spacelike boundaries; but this also is trivial, as "f preserves spacelike boundaries" is a statement about $\hat{f}$, and $\hat{\hat{f}} = \hat{f}$. $\square$

2.9 Theorem. *The future completion functor* $\hat{\ }$: **PdisPregSpbdFtopChron** $\to$ **FcplPdisPregSpbdFtopChron** *and the standard future injections* $\hat{\iota}_X : X \to \hat{X}$ *yield a left adjoint to the forgetful functor, hence a categorically unique way of providing a future completion, in a topological category respecting the* $\hat{\ }$*-topology, for past-distinguishing, regular chronological sets with spacelike boundaries.*

Proof. By Corollary 2.5, the map $\hat{\iota}_X : X \to \hat{X}$ is $\hat{\ }$-continuous, and it clearly preserves spacelike boundaries (since $\hat{\hat{\iota}}_X : \hat{X} \to \hat{\hat{X}} = \hat{X}$ is the identity). Thus, $\hat{\iota}_X$ is in the category **PdisPregSpbdFtopChron**, yielding a natural transformation $\hat{\boldsymbol{\iota}} : \mathbf{I} \dot{\to} \mathbf{U} \circ \hat{\ }$, where $\mathbf{I}$ is the identity functor on **PdisPregSpbdFtopChron** and $\mathbf{U}$ is the forgetful functor from **FcplPdisPregSpbdFtopChron** to **PdisPreg-SpbdFtopChron**; what this says is that for any $f : X \to Y$ in **PdisPregSpbd-FtopChron**, $\hat{\iota}_Y \circ f = \hat{f} \circ \hat{\iota}_X$ (part of Proposition 6 in [H]).

All that is needed to establish the left adjunction is the universality property: For any $f : X \to Y$ in **PdisPregSpbdFtopChron**, with Y also future-complete, $\hat{f} : \hat{X} \to Y$ is the unique future-continuous, $\hat{\ }$-continuous map, preserving spacelike boundaries, such that $\hat{f} \circ \hat{\iota}_X = f$. The unique existence of $\hat{f}$ for being future-continuous and satisfying this condition is Corollary 7 in [H]; that $\hat{f}$ is also $\hat{\ }$-continuous and preserves spacelike boundaries is Corollary 2.8 just above. Standard category theory (as in [M]) then gives the categorical uniqueness of any left adjoint (i.e., any other functor left-adjoint to the forgetful functor would be naturally equivalent to $\hat{\ }$; a natural equivalence is a natural transformation made up of isomorphisms). $\square$

Upon application of this functor to one of the objects most of interest—a strongly causal spacetime M—the resultant $\hat{M}$ is actually M^+, i.e., the GKP Future Causal Boundary construction (defined as the past-determination of $\hat{M}$); this is because, as observed above, a spacetime with only spacelike boundaries is already past-determined (as is, therefore, its future chronological completion). However, for categorical results, we cannot identify $\hat{\ }$ with $+$, since the requisite category contains objects which are not past-determined. For categorical discussion of the GKP operation, we must pursue a discussion of past-determination, the subject of Section 3.

3. Regular Chronological Sets and Past Determination

The past-determination functor $\mathbf{p}$: **PregChron** $\to$ **PdetPregChron** is the categorical method for making a regular chronological set past-determined; it adds some additional chronology relations, in case the original object is not past-determined already. Specifically, if X is a chronological set with chronology relation $\ll$, then X^p is the same set with chronology relation $\ll^p$ holding between x and y if either $x \ll y$ or if $I^-(x)$ is non-empty and $I^-(x) \subset I^-(w)$ for some $w \ll y$. For any $f : X \to Y$, $f^p : X^p \to Y^p$ is the same set-function; for any X, $\iota^p_X : X \to X^p$ is the identity on the set-level.

Propositions 10 and 11 and Corollary 12 in [H] establish these facts: For any chronological set X, X^p is past-determined; if X is past-determined, then $X^p = X$; if X is past-distinguishing or future-complete, so is X^p. For any future-continuous $f : X \to Y$, so long as X is regular, f^p is also future-continuous; thus $\mathbf{p}$ is a functor as claimed above. Within the past-regular category, the maps ι^p_X are future-continuous and form a natural transformation $\boldsymbol{\iota}^p : \mathbf{I} \dot{\to} \mathbf{U} \circ \mathbf{p}$, where $\mathbf{I}$ is the identity

functor on **PregChron** and $\mathbf{U}$: **PdetPregChron** $\to$ **PregChron** is the forgetful functor. The functor $\mathbf{p}$ and the natural transformation $\boldsymbol{\iota}^{\mathbf{p}}$ form a left adjoint to the forgetful functor $\mathbf{U}$ in virtue of the universality property: For any $f : X \to Y$ in **PregChron** with Y past-determined, $f^p : X^p \to Y$ is the unique future-continuous map with $f^p \circ \iota^p_X = f$.

Our first task is to see that all this machinery carries over to the topological categories: We need that past-determination preserves objects in the right categories, that functions stay continuous (with respect to the $\hat{\ }$-topologies) upon past-determination, and that the maps ι^p_X are continuous. This all falls in line quite nicely; the crucial fact is that past-determination has no effect on the $\hat{\ }$-topology.

We need a lemma to identify the IPs in X^p with those in X, analogous to the lemma used in Theorem 2.4: Let $\mathcal{IP}(X)$ denote the set of all IPs of a chronological set X. Given a chronological set X, for any $P \in \mathcal{IP}(X)$, let P^p denote $I^{-p}[P]$, where I^{-p} denotes the past using the $\ll^p$ relation; and for any $Q \in \mathcal{IP}(X^p)$, let Q_0 denote $I^-[Q]$ (I^- denoting past using $\ll$).

Lemma 3.1. *For any chronological set X, the maps $P \mapsto P^p$ and $Q \mapsto Q_0$ establish an isomorphism, as partially ordered sets under inclusion, between $\mathcal{IP}(X)$ and $\mathcal{IP}(X^p)$.*

Proof. First note that for any x, y, z in X, $x \ll y \ll^p z$ implies $x \ll z$ (because $I^-(y) \subset I^-(w)$ for some $w \ll z$ and $x \in I^-(y)$, so $x \in I^-(w)$, i.e., $x \ll w$); and $x \ll^p y \ll z$ implies $x \ll^p z$ (because $I^-(x)$ is non-empty and $I^-(x) \subset I^-(w)$ for some $w \ll y$, hence $w \ll z$). Conversely, $x \ll z$ implies there is some y with $x \ll y \ll^p z$ (there is some y with $x \ll y \ll z$, and $y \ll z$ implies $y \ll^p z$); and $x \ll^p z$ implies there is some y with $x \ll^p y \ll z$ ($I^-(x) \subset I^-(w)$ for some $w \ll z$; pick y with $w \ll y \ll z$). It follows that for any $z \in X$, $I^-[I^{-p}(z)] = I^-(z)$ and $I^{-p}[I^-(z)] = I^{-p}(z)$; therefore, for any $A \subset X$, $I^-[\,I^{-p}[A]\,] = I^-[A]$ and $I^{-p}[\,I^-[A]\,] = I^{-p}[A]$.

In particular, for any $P \in \mathcal{IP}(X)$, let c be a future chain generating P; then $P^p = I^{-p}[P] = I^{-p}[\,I^-[c]\,] = I^{-p}[c]$. Therefore, since c is also a future chain in X^p, we have that P^p is in $\mathcal{IP}(X^p)$. Also, for any $Q \in \mathcal{IP}(X^p)$, let c be a future chain in X^p generating Q, and let c' be an associated chain in X; then $Q_0 = I^-[Q] = I^-[\,I^{-p}[c]\,] = I^-[\,I^{-p}[c']\,] = I^-[c']$. Therefore, Q_0 is in $\mathcal{IP}(X)$. Thus, the two maps at least have targets as advertised.

We have, for any $P \in \mathcal{IP}(X)$, $(P^p)_0 = I^-[\,I^{-p}[P]\,] = I^-[P] = P$; and for any $Q \in \mathcal{IP}(X^p)$, $(Q_0)^p = I^{-p}[\,I^-[Q]\,] = I^{-p}[Q] = Q$. Thus, the the two maps yield a bijection of the respective sets of IPs. The preservation of the subset relation is evident. $\square$

Note that X is regular if and only if X^p is regular: If $I^-(x) = P \cup P'$ for P and P' in $\mathcal{IP}(X)$, neither equal to $I^-(x)$, then $I^{-p}(x) = I^{-p}[I^-(x)] = I^{-p}[P \cup P'] = I^{-p}[P] \cup I^{-p}[P'] = P^p \cup (P')^p$, neither equal to $I^{-p}(x)$ (if, e.g., $P^p = I^{-p}(x)$, then $P = (P^p)_0 = I^-[P^p] = I^-[I^{-p}(x)] = I^-(x)$). The other direction follows in exactly analogous fashion.

Proposition 3.2. *For any regular chronological set X, $\iota^p_X : X \to X^p$ is a homeomorphism in the respective $\hat{\ }$-topologies.*

Proof. Let L and L^p denote the limit-operators for, respectively, X and X^p. Let σ be any sequence in X with $x \in L(\sigma)$. For all $y \ll^p x$, there is some z with

$y \ll^p z \ll x$. For all n sufficiently large, $z \ll \sigma(n)$; then $y \ll^p z \ll \sigma(n)$, so $y \ll^p \sigma(n)$. For any IP Q of X^p with $Q \supset I^{-p}(x)$, Lemma 3.1 gives us $Q_0 \supset (I^{-p}(x))_0 = I^-(x)$. Suppose for every $y \in Q$, there is some subsequence $\tau \subset \sigma$ with, for all n, $y \ll^p \tau(n)$. For any $z \in Q_0$, there is some $y \in Q$ with $z \ll y$; then for some subsequence $\tau \subset \sigma$, for all n, $y \ll^p \tau(n)$. This gives us $z \ll y \ll^p \tau(n)$, so $z \ll \tau(n)$ for all n. Therefore, $Q_0 = I^-(x)$. By Lemma 3.1, $Q = I^{-p}(x)$. This shows $x \in L^p(\sigma)$.

Now let σ be any sequence with $x \in L^p(\sigma)$. A formally identical proof, with $\ll$, I^-, and the $(\ )^p$ map swapped, respectively, with $\ll^p$, I^{-p}, and the $(\ )_0$ map, establishes that $x \in L(\sigma)$.

With identical limit-operators, X and X^p have the same $\hat{\ }$-topologies; specifically, the identity map (ι^p_X) is a homeomorphism. $\square$

Theorem 3.3. *Past determination is a functor* $\mathbf{p} : \mathbf{PregFtopChron} \to \mathbf{PdetPregFtopChron}$*; the maps* ι^p_X *form a natural transformation* $\boldsymbol{\iota}^{\mathbf{p}}$ *in the* **PregFtop** *categories; and* $\mathbf{p}$ *and* $\boldsymbol{\iota}^{\mathbf{p}}$ *form a left adjoint for the forgetful functor* $\mathbf{U} : \mathbf{PdetPregFtopChron} \to \mathbf{PregFtopChron}$.

All this is also true in the **SpbdFtop** *categories.*

Proof. To show that $\mathbf{p}$ is a functor for the **Ftop** categories, we need only show that for $f : X \to Y$ future-continuous and $\hat{\ }$-continuous, that $f^p : X^p \to Y^p$ is $\hat{\ }$-continuous. This follows immediately from Proposition 3.2 and the commutative diagram $f^p \circ \iota^p_X = \iota^p_Y \circ f$. To show $\boldsymbol{\iota}^{\mathbf{p}}$ is a natural transformation for the topological categories, we just need each ι^p_X $\hat{\ }$-continuous, i.e., Proposition 3.2. Then the universality property for the chronological categories translates immediately into the appropriate universality property for the topological categories—for any future-continuous, $\hat{\ }$-continuous $f : X \to Y$ with X regular and Y regular and past-determined, $f^p : X^p \to Y$ is the unique future-continuous and $\hat{\ }$-continuous map satisfying $f^p \circ \iota^p_X = f$.

To extend these results to the **SpbdFtop** categories, we need to show that $\widehat{\iota^p_X}$ preserves inobservables, and if $\hat{f}$ preserves inobservables, then so does $\widehat{f^p}$:

Since the maps of Lemma 3.1 between $\mathcal{IP}(X)$ and $\mathcal{IP}(X^p)$ preserve inclusion among IPs, they also preserve the property of being inobservable or not. For $P \in \hat{\partial}(X)$, $\widehat{\iota^p_X}(P) = P^p$ (reason: $I^{-p}[\,\iota^p_X[P]\,] = I^{-p}[P] = P^p$, which is in $\hat{\partial}(X^p)$, so that is $\widehat{\iota^p_X}(P)$). Thus, $\widehat{\iota^p_X}$ preserves inobservables in $\hat{\partial}(X)$. For $x \in X$, if $I^{-p}(\iota^p_X(x)) = I^{-p}(x) \subset Q$ for some $Q \in \mathcal{IP}(X^p)$, then $I^-(x) = I^-[I^{-p}(x)] \subset I^-[Q] = Q_0 \in \mathcal{IP}(X)$; therefore, ι^p_X preserves inobservables in X.

For $f : X \to Y$, with $\hat{f} : \hat{X} \to \hat{Y}$ preserving inobservables, consider $\widehat{f^p} : \widehat{X^p} \to \widehat{Y^p}$. First note that any $x \in X$ is inobservable in X if and only if it is inobservable in X^p (if $I^-(x) \subset P$, then $I^{-p}(x) = I^{-p}[I^-(x)] \subset I^{-p}[P] = P^p$; other directions similarly). Thus, if $x \in X^p$ is inobservable in X^p, then it's also inobservable in X, so $f(x)$ is inobservable in Y, so $f^p(x) = f(x)$ is inobservable in Y^p. Second, consider $Q \in \hat{\partial}(X^p)$: If Q is inobservable, then so is Q_0, so $\hat{f}(Q_0)$ is also inobservable. Now suppose $\hat{f}(Q_0) = y \in Y$, i.e., that $I^-[\,f[Q_0]\,] = I^-(y)$. Let Q_0 be generated by a future chain c, which then also generates Q (in X^p); then $I^{-p}[\,f^p[Q]\,] = I^{-p}[\,f^p[c]\,] = I^{-p}[\,f[c]\,] = I^{-p}[\,f[Q_0]\,] = I^{-p}[I^-[\,f[Q_0]\,]] = I^{-p}[I^-(y)] = I^{-p}(y)$, so $\widehat{f^p}(Q) = y$ also. Thus, $\widehat{f^p}(Q)$ is inobservable. If, on the other hand, $\hat{f}(Q_0) \in \hat{\partial}(Y)$, i.e., $I^-[\,f[Q_0]\,]$ is not any $I^-(y)$, then for Q_0 generated

by c, $\hat{f}(Q_0) = I^-[\,f[c]\,]$. Then $I^{-p}[\,f^p[Q]\,] = I^{-p}[\,f[Q]\,] = I^{-p}[\,f[c]\,]$ is no $I^{-p}(y)$, for then $I^-(y) = I^-[I^{-p}(y)] = I^-[\,I^{-p}[\,f[c]\,]\,] = I^-[\,f[c]\,] = \hat{f}(Q_0)$, which we are supposing cannot be any $I^-(y)$. Therefore, $I^{-p}[\,f^p[Q]\,]$ is some $P \in \hat{\partial}(Y^p)$ (specifically, $P = I^{-p}[\,f[c]\,]$) and $\widehat{f^p}(Q) = P$. If $P \subset P'$ for some $P' \in \hat{\partial}(Y^p)$, then $P_0 \subset P'_0$, and $P_0 = I^-[\,I^{-p}[\,f[c]\,]\,] = I^-[\,f[c]\,] = \hat{f}(Q_0)$. Since this last is inobservable, we must have $P'_0 = P_0$, so $P' = P$. This says, again, that $\widehat{f^p}(Q)$ is inobservable. $\square$

We thus have a functor $\hat{}\circ\mathbf{p}$: **PdisPregSpbdFtopChron** $\to$ **FcplPdetPdisPregSpbdFtopChron** and a natural transformation $\hat{\boldsymbol{\iota}}\circ\boldsymbol{\iota}^{\mathbf{p}}$, forming a left adjoint to the forgetful functor. But to reach the actual GKP construction we need the functor $+$, where $X^+ = (\hat{X})^p$. Recall that this functor is constructed using $j_X : \widehat{X^p} \to (\hat{X})^p$ via $f^+ = j_Y \circ \widehat{f^p} \circ (j_X)^{-1} : X^+ \to Y^+$ (for $f : X \to Y$); and the same maps are also used to form the natural transformation $\boldsymbol{\iota}^+$ via $\iota^+_X = j_X \circ \hat{\iota}_{X^p} \circ \iota^p_X : X \to X^+$. The action of j_X on the elements of $\hat{\partial}(X^p)$ is just the map $Q \mapsto Q_0$ of Lemma 3.1.

We need to show that j_X and $(j_X)^{-1}$ are $\hat{}$-continuous. But this follows automatically, since j_X is an isomorphism of the chronological sets (Proposition 13 of [H]), and the $\hat{}$-topologies are constructed directly from the respective chronology relations. We also need that $\widehat{j_X}$ and its inverse preserve inobservables; this follows automatically for the same reason. Thus, all the elements are in order:

Theorem 3.4. *There is a functor $+$ from* **PdisPregSpbdFtopChron** *to* **FcplPdetPdisPregSpbdFtopChron**, *with a natural transformation $\boldsymbol{\iota}^+$ forming a left adjoint to the forgetful functor.* $\square$

This completes the categorical formulation of the Future Chronological Boundary in the topological, spacelike-boundary category. But we can also state results that are not strictly categorical, such as those that involve objects in different categories (i.e., obeying different hypotheses). For instance, From Theorem 14 in [H] we obtain this result:

Theorem 3.5. *Let X and Y be regular chronological sets with only spacelike boundaries, with Y past-distinguishing. For any future-continuous, $\hat{}$-continuous map $f : X \to Y$, preserving spacelike boundaries, there is a unique future-continuous, $\hat{}$-continuous map $f^+ : X^+ \to Y^+$, preserving spacelike boundaries, that commutes with the natural transformation $\boldsymbol{\iota}^+$.*

Proof. Theorem 14 of [H] gives the unique existence of f^+ with $f^+ \circ \iota^+_X = \iota^+_Y \circ f$, just for f future-continuous. Y does not need to be regular for that, but it appears in the hypotheses here, as we have not yet defined $\hat{}$-topology for non-regular chronological sets.

Theorems 2.7 and 3.3, respectively, give us $\hat{}$-continuity for the the future completion and past-determination of future-continuous functions. Theorem 3.3 requires no more than the hypotheses given here, but Theorem 2.7 is stated for the domain being past-distinguishing, as well as regular. However, examination of the proof there shows that past-distinguishment is never employed: It is in the hypotheses for the target-space only so as to ensure that $\hat{f} : \hat{X} \to \hat{Y}$ can be defined, and is in the hypotheses of the domain only so that the theorem is stated in terms of a functor on a category. Thus, we already have all we need to know that f^+ here is $\hat{}$-continuous. $\square$

More along these lines can be accomplished once we admit non-regular chronological sets into our consideration in Section 4 (see Proposition 4.6).

Theorem 1.1 yields an obvious topological result, not requiring spacelike boundaries:

Theorem 3.6. *Let X be any regular chronological set; then any future-completing, past-distinguishing boundary on X is $\widehat{\ }$-homeomorphic to $\hat{\partial}(X)$.*

Proof. If Y is a past-distinguishing future completion of X in the sense of Theorem 1.1 (with $i : X \to Y$), then that theorem yields a chronological isomorphism i^+ of $X^+ = (\hat{X})^p$ with $Y^+ = Y^p$. This map carries X in $\hat{X}$ onto $Y_0 = i[X]$ in Y; hence, it carries $\hat{\partial}(X)$ onto $\partial(Y) = Y - Y_0$. Since the $\widehat{\ }$-topologies are determined solely by the chronology relations (in this case, the past-determined chronology relations), this is also a $\widehat{\ }$-homeomorphism. In particular, we get a homeomorphism of $\hat{\partial}(X)$ with $\partial(Y)$ in the $\widehat{\ }$-topologies determined by $\ll^p$ in $\hat{X}$ and Y. But by Proposition 3.2, there is no difference in the $\widehat{\ }$-topologies determined by $\ll^p$ and by $\ll$. Thus, we have a homeomorphism of the two boundaries in the original spaces, $\hat{X}$ and Y. $\square$

Again, this strong rigidity will be relaxed once we consider non-regular chronological sets, allowing for a generalized sense of future completion (see Corollary 4.9).

If we don't insist on the future boundary actually future-completing the chronological set we start out with, then we must have a portion of the Future Chronological Boundary, at least $\widehat{\ }$-topologically (the chronology can be different, but it is the same in the past-determination). This can be couched as a semi-rigid version of Theorem 2.4, with the addition of past-distinguishing:

Theorem 3.7. *Let $\bar{X}$ be a past-distinguishing, regular chronological set with X a subset of $\bar{X}$ satisfying the following:*

(1) *The restriction of $\ll$ to X yields another regular chronological set; and*
(2) *for any $p \ll q$ in $\bar{X}$, there is some $x \in X$ so that $p \ll x \ll q$.*

Then there is a topological embedding of $\bar{X}$ into $\hat{X}$, which is the identity on X.

Proof. Let I^- denote the past in X, $\bar{I}^-$ the past in $\bar{X}$, and $\hat{I}^-$ the past in $\hat{X}$. Let $\bar{\partial}(X)$ denote $\bar{X} - X$, our (partial) future boundary on X.

Consider $p \in \bar{\partial}(X)$, and let $Q = \bar{I}^-(p)$. Apply the operation from the Lemma in Theorem 2.4 to obtain $Q_0 = Q \cap X$. Suppose there were $x \in X$ such that $Q_0 = I^-(x)$; then $\bar{I}^-(x) = \bar{I}^-[Q_0] = \overline{(Q_0)} = Q = \bar{I}^-(p)$; then past-distinguishing yields $x = p$, which cannot be. Therefore, $Q_0 \in \hat{\partial}(X)$.

Thus, we can define a map $i : \bar{X} \to \hat{X}$ by $i(x) = x$ for $x \in X$ and $i(p) = (\bar{I}^-(p))_0$ for $p \in \bar{\partial}(X)$, and this is the identity on X, taking boundary points to boundary points. This map is injective, since $i(p) = i(p')$ means $(\bar{I}^-(p))_0 = (\bar{I}^-(p'))_0$, hence $\overline{(\bar{I}^-(p))_0} = \overline{(\bar{I}^-(p'))_0}$, hence $\bar{I}^-(p) = \bar{I}^-(p')$, so $p = p'$ by past-distinguishing.

That i is chronological is very easy to establish. But we need more: We want a chronological isomorphism onto the image. This, however, is not necessarily true for i; but it is true for the past-determination, $i^p : \bar{X}^p \to (\hat{X})^p$. The only real difficulty comes with $i(p) \ll^p x$ for $p \in \bar{\partial}(X)$ and $x \in X$; we need to obtain $p \ll^p x$:

Knowing $i(p) \ll^p x$, we can find (as in the proof of Lemma 3.1) $q \in \hat{X}$ with $i(p) \ll^p q \ll x$. Then, whether q is in X or in $\hat{\partial}(X)$, there is $x' \in X$ with $q \ll x' \ll x$

(since for $q \in \hat{\partial}(X)$, $q \ll x$ means there is $w \ll x$ in X with $I^-(q) \subset I^-(w)$). Then we have $i(p) \ll^p x'$, so for some $w \ll x'$ in $\hat{X}$, $\hat{I}^-(i(p)) \subset \hat{I}^-(w)$. Let $Q = \bar{I}^-(p)$, so $i(p) = Q_0$. Then $\hat{I}^-(Q_0)$ includes all the elements of the IP Q_0. Since $\hat{I}^-(Q_0) \subset \hat{I}^-(w)$, it follows that $Q_0 \subset \hat{I}^-(w) \cap X \subset I^-(x')$. Therefore, $Q = \overline{(Q_0)} \subset \overline{I^-(x)} = \bar{I}^-(x')$. That, together with $x' \ll x$, gives us $p \ll^p x$.

All the other implications follow readily, yielding that i^p is a chronological isomorphism onto its image. This gives us i^p as a $\widehat{\ }$-homeomorphism onto its image. Finally, Proposition 3.2 gives us that i is also a $\widehat{\ }$-homeomorphism onto its image. $\square$

4. Topology for Non-Regular Chronological Sets

The example cited at the beginning of Section 2 illustrates both the desirability and the challenge of non-regular boundary elements for a spacetime: In that example, the spacetime M, consisting of $\mathbb{L}^2$ with the negative time-axis deleted, has, as Future Chronological Boundary, two closed half-lines, representing either side of the missing axis. This yields the rather eccentric situation of a pair of non-Hausdorff elements in the boundary: The two future-ends of those half-lines, P_0^+ and P_0^-, each trying to enact the role of the missing point $(0,0)$, are both limits of the sequence $\sigma(n) = (0, 1/n)$, i.e., both are in $L(\sigma)$. The solution to this somewhat unnatural construction, as provided in the full Causal Boundary of [GKP], is to meld the two points together, into the single boundary point B_0; the future of B_0 is defined to be $I^+((0,0))$ (plus the elements of the Future Causal Boundary "at infinity"), and its past is defined to be $I^-((0,0))$ except for the deleted semi-axis (plus the other boundary elements P_s^+ and P_s^- for $s < 0$; and, in the full Causal Boundary, the points "at infinity" of the Past Causal Boundary). This produces a chronological set, but $I^-(B_0)$ is decomposable as $\bar{P}_0^+ \cup \bar{P}_0^-$ (the bar denoting inclusion of the other boundary elements, as appropriate). This is, in some sense, a much more natural object to use as a boundary of M than is $\hat{\partial}(M)$. The challenge is to topologize it in a way that makes B_0 the (unique) limit of the sequence σ; B_0 should also be the limit of such sequences as $\tau^+(n) = (1/n, 0)$ and $\tau^-(n) = (-1/n, 0)$, and even of τ^0 defined by $\tau^0(2n) = \tau^+(n)$ and $\tau^0(2n+1) = \tau^-(n)$ (see Figure 8).

To explore this topology, we must make explicit use of those portions of the past of a point x which make $I^-(x)$ decomposable: For any point x in a chronological set X, consider $\mathcal{IP}_x = \{P \in \mathcal{IP}(X) \mid P \subset I^-(x)\}$ as a partially ordered set under inclusion. Call any $P \in \mathcal{IP}_x$ a *past component* of x if P is a maximal element of $\mathcal{IP}_x$. IPs are required to be non-empty, so $I^-(x)$ must be non-empty if x is to have any past components. So long as that condition is met, there will be a past component containing every point in its past:

Proposition 4.1. *Let X be a chronological set and x a point of X. For every $y \ll x$, there is a past component Q of x containing y.*

Proof. First note that the union of any chain of IPs is an IP: Recall that a past set P is an IP if and only if for any x_1 and x_2 in P, there is some $z \in P$ with $x_1 \ll z$ and $x_2 \ll z$. If $\{P_\alpha \mid \alpha < \Gamma\}$ is a chain of IPs, ordered by inclusion (Γ an ordinal number), then $P = \bigcup_{\alpha<\Gamma} P_\alpha$ is clearly a past set, and by the criterion just mentioned, it is an IP (for $x_1 \in P_\alpha$ and $x_2 \in P_\beta$, both x_1 and x_2 are in P_γ for

Therefore, every chain in $\mathcal{IP}_x$ has an upper bound. By Zorn's Lemma, every element of $\mathcal{IP}_x$ is contained in a maximal element of $\mathcal{IP}_x$, i.e., a past component of x.

For $y \ll x$, there is some y_1 with $y \ll y_1 \ll x$. We can define a future chain $\{y_n\}$ thus, by choosing y_{n+1} such that $y_n \ll y_{n+1} \ll x$. Then $P = I^-[\{y_n\}]$ is an element of $\mathcal{IP}_x$ containing y. P sits in a past-component Q of x, and we have $y \in P \subset Q$. $\square$

We next define a notion of a sequence "converging" to an IP, expressed in terms of a function $\mathcal{L} : \mathcal{S}(X) \to \mathfrak{P}(\mathcal{IP}(X))$ from sequences in X to the power set of IPs in X:

Limit-operator for IPs in a chronological set. For X any chronological set, the limit-operator $\mathcal{L}$ for IPs is defined thus:

For a sequence σ in X and $P \in \mathcal{IP}(X)$, $P \in \mathcal{L}(\sigma)$ if and only if

(1) for all $x \in P$, eventually $x \ll \sigma(n)$, and
(2) for any IP $Q \supset P$, if for all $x \in Q$, there is some subsequence $\tau \subset \sigma$ such that for all k, $x \ll \tau(k)$, then $Q = P$.

(Clause (2) is equivalent to this: for any IP Q properly containing P, for some $x \in Q$, eventually $x \not\ll \sigma(n)$.)

Thus, $\mathcal{L}$ plays the same role for IPs that the limit-operator L played before for points. In the example above, we have $\bar{P}_0^+$ is in $\mathcal{L}(\sigma)$ and in $\mathcal{L}(\tau^+)$, but not in $\mathcal{L}(\tau^-)$ or in $\mathcal{L}(\tau^0)$; and dually for $\bar{P}_0^-$. Our goal is to have all four sequences converge to B_0, whose past components are $\bar{P}_0^+$ and $\bar{P}_0^-$. It is especially the sequence τ^0, bouncing back and forth from τ^+ to τ^-, that motivates this definition:

Limit-operator for points in a chronological set. For X any chronological set, the limit-operator L for points is defined thus:

For σ a sequence in X and $x \in X$, $x \in L(\sigma)$ if and only if for every subsequence $\tau \subset \sigma$, there is a subsubsequence $\rho \subset \tau$ and a past component P of x, such that $P \in \mathcal{L}(\rho)$.

(We shall have frequent occasion to mention the family of past components of a point x; call this $\mathfrak{P}_x$.)

Thus, in the example at hand, we have $B_0 \in L(\tau^0)$ because for any subsequence $\tau' \subset \tau^0$, either τ' has infinitely many points in τ^+ or it has infinitely many in τ^- (or both). If the former, then it has a subsubsequence ρ^+ which lies in τ^+, and $\bar{P}_0^+ \in \mathcal{L}(\rho^+)$ (because $\bar{P}_0^+ \in \mathcal{L}(\tau^+)$); and if the latter, then we find a subsubsequence $\rho^- \subset \tau^-$ and $\bar{P}_0^- \in \mathcal{L}(\rho^-)$.

Note first of all that this definition of limit-operator reduces to the original one given in Section 2—call it L_0— for those points of X which are regular: For such x, this definition says that $x \in L(\sigma)$ if and only if, for every $\tau \subset \sigma$, there is a $\rho \subset \tau$ such that $I^-(x) \in \mathcal{L}(\rho)$; since $\sigma_1 \subset \sigma_2$ implies $\mathcal{L}(\sigma_2) \subset \mathcal{L}(\sigma_1)$, this is equivalent to saying that for all $\sigma' \subset \sigma$, $I^-(x) \in \mathcal{L}(\sigma')$. That is equivalent to saying that for all $\sigma' \subset \sigma$, $x \in L_0(\sigma')$. But that last is equivalent to saying $x \in L_0(\sigma)$.

Next note that this definition of limit-operator does, indeed, define a topology: If $\sigma' \subset \sigma$ and $x \in L(\sigma)$, then for any $\tau \subset \sigma'$, we also have $\tau \subset \sigma$, so there is $\rho \subset \tau$ and a past component P of x with $P \in \mathcal{L}(\rho)$; thus, $x \in L(\sigma')$ also. We will again call this the $\widehat{\ }$ topology.

Note that for any $x \in X$, with $\hat{x}$ denoting the constant sequence $\hat{x}(n) = x$, $\mathcal{L}(\hat{x}) = \mathfrak{P}_x$, the set of past components of x: To say that for all $y \in P$, $y \ll \hat{x}(n)$ (whether for all n large enough or for some subsequence of integers) is just to say that $P \subset I^-(x)$. Thus, the first clause in the definition of $\mathcal{L}(\hat{x})$ says $P \subset I^-(x)$, and the second says that for any IP $Q \supset P$, if $Q \subset I^-(x)$ also, then $Q = P$; and that is precisely the definition of a past component of x.

What we're aiming for is an analogue of Proposition 2.1, to show that points are closed. But we need the generalization of past-distinguishing from [H] for this: Call a chronological set X *generalized past-distinguishing* if for all x and y in X, if x and y share a past component—i.e., if $\mathfrak{P}_x \cap \mathfrak{P}_y \neq \emptyset$—then $x = y$.

Proposition 4.2. *For any generalized past-distinguishing chronological set X, for any $x \in X$, $L(\hat{x}) = \{x\}$; thus, all points are closed in the $\hat{\ }$-topology.*

Proof. We have $y \in L(\hat{x})$ if and only if there is a past component of y in $\mathcal{L}(\hat{x})$, i.e., $\mathfrak{P}_y \cap \mathfrak{P}_x \neq \emptyset$. $\square$

There are other notions that need to be generalized for non-regular chronological sets. For instance, in the example above, we have B_0 is the $\hat{\ }$-limit of the future chain $c = \{(1/n, -2/n)\}$, but it's not the future limit of c, since $I^-(B_0)$ includes both $\bar{P}_0^+$ and $\bar{P}_0^-$, while $I^-[c]$ is just the one past component $\bar{P}_0^+$. Thus, we are led to this definition (also from [H]): For c a future chain in a chronological set X, a point x is a *generalized future limit* of x if $I^-[c]$ is a past component of x.

Then we have an analogue of Proposition 2.2:

Proposition 4.3. *Let $c = \{x_n\}$ be a future chain in a chronological set X. Then a point x is a generalized future limit for c if and only if it is a $\hat{\ }$-limit for c; furthermore, if X is generalized past-distinguishing, then $L^\Omega[c] = L[c]$.*

Proof. Suppose x is a generalized future limit of c, so $P = I^-[c] \in \mathfrak{P}_x$. Then for all $y \in P$, eventually $y \ll x_n$. Also, for any IP $Q \supset P$, if for all $y \in Q$, $y \ll x_{n_k}$, then we also have for all $y \in Q$, $y \in I^-[c] = P$, so $Q = P$. Therefore, $P \in \mathcal{L}(c)$. For any subchain $c' \subset c$, we also have $P \in \mathcal{L}(c')$. Therefore, $x \in L(c)$.

Suppose $x \in L(c)$. For any subchain $c' \subset c$, there is a subsubchain $c'' \subset c'$ and a past component P of x with $P \in \mathcal{L}(c'')$. But for a future chain c, the elements of $\mathcal{L}(c)$ are the same as those of $\mathcal{L}(c')$ for any subchain c'; so the previous statement amounts just to saying that there is a past component P of x with $P \in \mathcal{L}(c)$. And that says that for all $y \in P$, eventually $y \ll x_n$, and for any IP $Q \supset P$, if for all $y \in Q$, $y \ll x_{n_k}$, then $Q = P$. But that just says that $P \subset I^-[c]$ and that P is a maximal IP for that property. Since $I^-[c]$ is an IP, that means $P = I^-[c]$, so x is a generalized future limit of c.

To show $L^\Omega[c] = L[c]$, we need only show that $L^2[c] = L[c]$ (which is $c \cup L(c)$). Consider any sequence of points σ in $L(c)$. From what we've just seen, each $\sigma(n)$ is a generalized future limit of c, i.e., has a past component which is $I^-[c]$. But by generalized past-distinguishment, there can be only a single generalized future limit of c, so this is a constant sequence $\hat{x}$, and $L(\hat{x}) = \{x\}$. $\square$

We can have an analogue of Theorem 2.4, showing how any sort of future boundary on a chronological set does, indeed, form a boundary in a reasonable topological sense. In Theorem 2.4, it was the insistence on (past) regularity that justified the word "future" as applied to the boundary; here, we will be content merely to have pasts of points non-empty:

Theorem 4.4. *Let $\bar{X}$ be a chronological set with every point having a non-empty past, and having a subset X satisfying the following:*

1. *The restriction of $\ll$ to X yields another chronological set with a non-empty past for each point; and*
2. *for any $p \ll q$ in $\bar{X}$, there is some $x \in X$ so that $p \ll x \ll q$.*

Then the $\widehat{\ }$-topology on X (as a chronological set in its own right) is the same as the subspace topology it inherits from the $\widehat{\ }$-topology on $\bar{X}$, and X is dense in $\bar{X}$.

Proof. Let $\bar{I}^-$, $\bar{L}$, $\bar{\mathcal{L}}$, and $\bar{\mathfrak{P}}$ denote the past operator, limit-operators, and past components in $\bar{X}$, the unbarred versions denoting the same in X. The Lemma of Theorem 2.4 is still valid, as it made no use of regularity: For $P \in \mathcal{IP}(X)$, $\bar{P} \in \mathcal{IP}(\bar{X})$, and for $Q \in \mathcal{IP}(\bar{X})$, $Q_0 \in \mathcal{IP}(X)$. In fact, for any $x \in X$, these maps yield a bijection between $\mathfrak{P}_x$ and $\bar{\mathfrak{P}}_x$: For P a past component of x, $\bar{P}$ is clearly a subset of $\bar{I}^-(x)$, and its maximality as such is provided by the preservation of inclusion by these maps; the other direction is similar.

We need to show that for any sequence σ in X and $x \in X$, $x \in L(\sigma)$ if and only if $x \in \bar{L}(\sigma)$.

Suppose $x \in L(\sigma)$. We must show that for every $\tau \subset \sigma$, there is a $\rho \subset \tau$ and a $Q \in \bar{\mathfrak{P}}_x$ with $Q \in \bar{\mathcal{L}}(\rho)$. We have $P \in \mathfrak{P}_x$ with $P \in \mathcal{L}(\rho)$. Then $Q = \bar{P}$ works: We already know $\bar{P} \in \bar{\mathfrak{P}}_x$. For any $p \in \bar{P} = \bar{I}^-[P]$, we can find $x \in P$ with $p \ll x$; then eventually $x \ll \rho(n)$, so $p \ll \rho(n)$. For any $Q' \in \mathcal{IP}(\bar{X})$ with $Q' \supset \bar{P}$, if for all $p \in Q'$, $p \ll \rho(n_k)$, then the same is true for Q'_0, and $Q'_0 \supset P$. Hence, $Q'_0 = P$, whence $Q' = \bar{P}$. Therefore, $\bar{P} \in \bar{\mathcal{L}}(\rho)$.

A formally identical proof, with barred and unbarred reversed, shows that if $x \in \bar{L}(\sigma)$, then $x \in L(\sigma)$ (note that for $Q \in \mathcal{IP}(\bar{X})$, $Q_0 = I^-[Q]$).

For density of X in $\bar{X}$, consider any $p \in \bar{X}$: It has a non-empty past, so it has a past component Q generated by a future chain c (Proposition 4.1). There is an interweaving chain c_0 which lies in X, generating Q_0. Then p is a generalized future limit of c_0 (in $\bar{X}$), so by Proposition 4.3, $p \in \bar{L}(c_0)$. $\square$

We have the immediate application to future completion, as in Corollary 2.5:

Corollary 4.5. *Let X be a chronological set with non-empty pasts for its points. Then the standard future injection $\hat{\imath}_X : X \to \hat{X}$ is a homeomorphism onto its image, and X is dense in $\hat{X}$.*

Proof. All we need do is observe that for all $P \in \hat{\partial}(X)$, the past of P in $\hat{X}$ is non-empty. $\square$

Applying future-completion to a non-regular chronological set is not actually a very satisfactory process, as the result is never generalized past-distinguishing, even if the original object is: Suppose X is generalized past-distinguishing and has a point x with distinct past components P and Q. Note that $P \in \hat{\partial}(X)$: If $P = I^-(y)$, then x and y share the past component P, so $x = y$, but we're assuming that $I^-(x) \neq P$. Thus, P is a point in its own right in $\hat{X}$. If P is generated by the future chain c in X, then $\bar{P} = \hat{I}^-[c]$ is the past of P in $\hat{X}$; but $\bar{P}$ is also a past component of x in $\hat{X}$, and $x \neq \bar{P}$. Therefore, $\hat{X}$ is not generalized past-distinguishing. In short: Forming $\hat{X}$ creates non-Hausdorff clusters of points associated with all the past components of any non-regular point in X.

A way around this can be found by instituting a new functor, generalized future completion: For any chronological set X, let $\hat{\partial}^g(X) = \{P \in \mathcal{IP}(X) \mid P$ is not a past

component of any point in $X\}$, the *Generalized Future Chronological Boundary* of X; and let $\hat{X}^g = X \cup \hat{\partial}^g(X)$, with chronology relation defined as for $\hat{X}$, the *generalized future completion* of X; among generalized past-distinguishing chronological sets, this amounts to a functor (into a category of generalized future-complete objects, a concept explained below), with $f : X \to Y$ giving rise to $\hat{f}^g : \hat{X}^g \to \hat{Y}^g$. In essence, $\hat{X}^g$ dispenses with the extraneous points that appear in $\hat{X}$, associated with each non-regular point in X, and interrelationships between the two functors $\widehat{\ }$ and $\widehat{\ }^{\mathbf{g}}$ can be explored (for example, there are maps $\pi_X : \hat{X} \to \hat{X}^g$—sending the past components of a non-regular point to that point—forming a natural transformation $\boldsymbol{\pi} : \widehat{\ } \dot{\to} \widehat{\ }^{\mathbf{g}}$). But there seems to be little point in pursuing a full categorical treatment of generalized future completion, as non-regular chronological sets are seldom the object upon which one wishes to construct a completing boundary; rather, the more usual course is to form a non-regular object as the intended completion of a starting object which is regular (e.g., a spacetime).

Probably the most common usage of non-regular chronological sets is the formation of boundaries for spacetimes like that of the example M above, where the Future Chronological Boundary yields an unsatisfying result: the non-Hausdorff pair of P_0^+ and P_0^-; in this case, the full GKP Causal Boundary results in the somewhat more natural boundary (with B_0)—at any rate, it's a Hausdorff boundary. It is not apparent what purpose is served by looking at the categorical operations applied to such constructs; for instance, if $\bar{M} = M \cup \bar{\partial}(M)$ is formed from $\hat{M}$ by replacing P_0^+ and P_0^- by B_0, then $\widehat{\bar{M}}$ gives us all three boundary points: $\bar{B}_0$ as part of $\bar{M}$, then $\bar{P}_0^+$ and $\bar{P}_0^-$ as IPs in $\bar{M}$, and it is unclear what use this object has.

None the less, it may be of use to pursue the categorical approach for non-regular chronological sets, if for no other purpose than to convince ourselves that the $\widehat{\ }$-topology in the non-regular case really does have appropriate properties.

Proceeding in this spirit, we have to generalize our sense of appropriate morphism in **Chron**: If we look at the projection map $\pi : \hat{M} \to \bar{M}$ (π is the identity on $\bar{M}$, save that $\pi(P_0^+) = \pi(P_0^-) = B_0$), we see it is not future-continuous: P_0^+ is the future limit of the future chain $c = \{(1/n, -2/n)\}$, but B_0 is not the future limit of c, as the past of B_0 contains both past components, not just the one that c is in. But B_0 *is* a generalized future limit of c. That is the key to the morphisms we need to be considering between non-regular chronological sets: As in [H], a chronological map $f : X \to Y$ between chronological sets will be called *generalized future-continuous* if for any future chain c in X with generalized future limit x, $f(x)$ is a generalized future limit of $f[c]$. Let **GChron** be the category whose objects are chronological sets with all points having a non-empty past, and with morphisms being generalized future-continuous maps. As in [H], we will call a chronological set *generalized future-complete* if every future chain has a generalized future limit. Finally, X will be called *generalized past-determined* if whenever a past component of x is contained in $I^-(w)$ for some $w \ll y$, we have $x \ll y$.

It is easy to see that the notions of generalized past-distinguishing and generalized past-determined are stronger notions than, respectively, past-distinguishing and past-determined, while generalized future-complete is a weaker notion than future-complete. (Note that a spacetime with only spacelike boundaries—or its future completion—is generalized past-determined: It is past-determined and also regular.) But generalized future-continuous is neither weaker nor stronger than future-continuous.

We also must generalize our notion of inobservability; in fact, we will redefine it, in a manner which includes the previous definition when applied to regular points: A point x in a chronological set is *inobservable* if for every past component P of x, the only IP which contains P is P itself. (Notions of spacelike boundaries and functions preserving spacelike boundaries are unchanged.)

Here is a collection of categorical and quasi-categorical results in the generalized categories. Generalized past-determination is utilized only in the purely chronological (i.e., not topological) category. For the topological category, the assumption of spacelike boundaries (necessary for the application of the techniques from Proposition 2.7) suffices in place of generalized past-determination. The second statement here leads to some further general categorical results. The third statement is notable for leading to the quasi-rigidity results in the immediately succeeding theorems, and its quasi-categorical nature will prove important for proving Theorem 5.3.

Proposition 4.6. *Let $f : X \to Y$ be a morphism in* **GChron***.*

(1) *Suppose Y is generalized past-determined and generalized past-distinguishing; then $\hat{f} : \hat{X} \to \hat{Y}$ is also in* **GChron** *and is the unique generalized future-continuous map such that $\hat{f} \circ \hat{\iota}_X = \hat{\iota}_Y \circ f$.*

(2) *Suppose that f, X, and Y are in* **SpbdFtopGchron** *with Y generalized past-distinguishing; then $\hat{f}$ is also in* **SpbdFtopGchron***.*

(3) *Suppose, in addition to the above, that Y is generalized future-complete; then there is a unique generalized future-continuous and $\hat{\ }$-continuous map $\tilde{f}^g : \hat{X} \to Y$ such that $\tilde{f}^g \circ \hat{\iota}_X = f$.*

Proof. (1) We only need Y to be past-determined and past-distinguishing to obtain the chronological map $\hat{f}$ (Proposition 6 in [H]). We need to show here that $\hat{f}$ is generalized future-continuous. But this is easy: If $x \in X$ is a generalized future limit of a future chain c in $\hat{X}$, then we can interpolate elements of X to obtain a chain c_0 in X, for which x is a generalized future limit; then $f(x)$ is a generalized future limit of $f[c_0]$ in Y, so it also is in $\hat{Y}$. If $P \in \hat{\partial}(X)$ is a generalized future limit of c, then P is generated by the chain c_0 (and, actually, P is an ordinary future limit of c), so $\hat{f}(P)$ is generated by $f[c_0]$ (and, actually, $\hat{f}(P)$ is an ordinary future limit of $f[c]$). And $\hat{f}$ is unique for a generalized future-continuous function obeying $\hat{f} \circ \hat{\iota}_X = \hat{\iota}_Y \circ f$, since $\hat{f}(P)$ is then forced to be a generalized future limit of $f[c_0]$, and there is only one such, since Y is generalized past-distinguishing (which implies the same for $\hat{Y}$).

(2) Now assume we're in the spacelike boundaries category: As in the proof of Proposition 2.7, to have $\hat{f}$ future-continuous, we don't need Y to be past-determined, since X has only spacelike boundaries; but we need to show $\hat{f}$ is $\hat{\ }$-continuous. The proof of Theorem 4.4 demonstrates that for any Z in **GChron**, for any sequence σ in X, $L_Z(\sigma) = L_{\hat{Z}}(\sigma) \cap Z$, just as was needed in the proof of Proposition 2.7. Then the formal properties of the limit-operator complete the portion of the proof showing that if $x \in X$ is in $L_{\hat{X}}(\sigma)$ for a sequence σ lying in X, then $f(x) \in L^{\Omega}_{\hat{Y}}[\, f[\sigma]\,]$.

Now consider the same sequence σ with $P \in \hat{\partial}(X)$ in $L_{\hat{X}}(\sigma)$. Then P is regular, as is $\hat{f}(P)$ (reason: Let c be a chain in X generating P, and let $Q = I^-[\, f[c]\,]$. If $Q \in \hat{\partial}(Y)$, then $\hat{f}(P) = Q$ and Q is regular (because it's in $\hat{\partial}(Y)$); while if $Q = I^-(y)$ for some $y \in Y$, then $\hat{f}(P) = y$ and y is regular (because its past

is the IP Q).). Therefore, the same proof as in Proposition 2.7 applies, showing $\hat{f}(P) \in L_{\hat{Y}}(f[\sigma])$.

Now consider the case of a sequence $\sigma = \{P_n\}$ lying in $\hat{\partial}(X)$, with $x \in X$ in $L_{\hat{X}}(\sigma)$. Let $\sigma' = \hat{f}[\sigma]$; we need to show that $f(x) \in L_{\hat{Y}}(\sigma')$, i.e., that for any $\tau' \subset \sigma'$, there is a $\rho' \subset \tau'$ and a past component Q of $f(x)$ with $Q \in \mathcal{L}_{\hat{Y}}(\rho')$. For any such subsequence τ' of σ', let τ be the corresponding subsequence of σ, i.e., so that $\hat{f}[\tau] = \tau'$; then there is a subsubsequence $\rho \subset \tau$ and a past component P of x such that $P \in \mathcal{L}_{\hat{X}}(\rho)$. Let c be a chain in X generating P, and let $Q = I^-[\,f[c]\,]$. We have x a generalized future limit of c, so $f(x)$ must be a generalized future limit of $f[c]$ (f being generalized future continuous); that says precisely that Q is a past component of $f(x)$. We know that for all m, eventually $c(m) \ll \rho(n)$; therefore, for all m, eventually $f(c(m)) \ll \hat{f}(\rho(n))$. Since x is a limit of boundary elements, which must be inobservable, x is also inobservable (inobservables in $\hat{X}$ being closed); therefore, $f(x)$ is also inobservable. Hence, we need not worry about any IP $Q' \supset Q$, as necessarily then $Q' = Q$. Therefore, $Q \in \mathcal{L}_{\hat{Y}}(\rho')$ for $\rho' = \hat{f}[\rho]$: We have $f(x) \in L_{\hat{Y}}(\sigma)$.

Next consider $P \in \hat{\partial}(X)$ in $L_{\hat{X}}(\sigma)$ for the same $\sigma = \{P_n\}$. Then P and $\hat{f}(P)$ are regular, and the proof of Proposition 2.7 applies.

For a general sequence σ, the same stratagem as used in Proposition 2.7 works here: If there is some subsequence $\sigma' \subset \sigma$ lying in $\hat{\partial}(X)$, then any $p \in L_{\hat{X}}(\sigma)$ is inobservable, and the proof above applies. And if there is no such subsequence, then we can ignore the finite number of elements of σ lying in $\hat{\partial}(X)$ and apply the other parts of the proof.

That $\hat{f}$ preserves spacelike boundaries follows precisely as in Corollary 2.8, so $\hat{f}$ is a morphism of **SpbdFtopGChron**.

(3) Finally, consider Y to be generalized future-complete. Theorem 15 of [H] yields a unique generalized future-continuous $\tilde{f}^g : \hat{X} \to Y$ with $\tilde{f}^g \circ \hat{\iota}_X = f$ ($\tilde{f}^g$ is defined thus: for $P \in \hat{\partial}(X)$ generated by a future chain c, $\tilde{f}^g(P)$ is the unique generalized future limit of $f[c]$); we just need to show it $\hat{\ }$-continuous. (In [H], this map was called $\hat{f}^g$, disrespecting the potential categorical usage of that symbolism. Using that symbolism properly, we can identify $\tilde{f}^g$ in terms of the functor and natural transformation alluded to above: $\tilde{f}^g = \hat{f}^g \circ \pi_X$.)

For a sequence σ lying in X, consider a point $x \in X$ within $L_{\hat{X}}(\sigma)$: As before, this means $x \in L_X(\sigma)$, whence, by $\hat{\ }$-continuity of f, $f(x) \in L^{\Omega}_Y(f[\sigma])$, as needed. Now consider $P \in \hat{\partial}(X)$ in $L_{\hat{X}}(\sigma)$: Then, P being regular, the proof in Proposition 2.7 applies with minor modification (for P generated by a chain c, $\tilde{f}^g(P)$ is in Y, even if $I^-[\,f[c]\,]$ is not $I^-(y)$ for any $y \in Y$; $I^-[\,f[c]\,]$ is a past component of a unique y, however, and that is $\tilde{f}^g(P)$). For a sequence σ lying in $\hat{\partial}(X)$ or, more generally, in $\hat{X}$, the same proof applies as in (2) above. $\square$

It may be instructive to examine a couple of examples, illustrating how the spacelike nature of the boundary is important for continuity of the induced map in statement (3), even for cases of simple inclusion, with an isomorphism onto the image:

First consider a case with spacelike boundary (see Figure 9): Let X be lower Minkowski half-space, $\mathbb{L}^3_- = \{(x, y, t) \mid t < 0\}$; then $\hat{X}$ is clearly $\{(x, y, t) \mid t \leq 0\}$ (actually, $(x, y, 0)$ refers to the obvious IP in X), and X manifestly has only space-

like boundaries and is past-distinguishing. We will let Y be derived from $\hat{X}$ by identification of some points: Fix attention on two curves in $\hat{X}$, $L^- = \{(x,-1,0) \mid x > 0\}$ and $L^+ = \{(x,1,0) \mid x > 0\}$. For each $x \geq 0$, let $p_x^- = (x,-1,0)$ and $p_x^+ = (x,1,0)$. Define the set Y as $\hat{X}/\sim$, where $\sim$ is the equivalence relation given by $p_x^- \sim p_x^+$ for each $x > 0$, and no other points identified together; thus, $\sim$ identifies the two open half-lines L^- and L^+ into a single line L. For each $x > 0$, let p_x denote the equivalence class of p_x^+ (or of p_x^-). Make Y into a chronological set by giving it the same chronology relation as on $\hat{X}$, with the proviso that $I_Y^-(p_x) = I_{\hat{X}}^-(p_x^-) \cup I_{\hat{X}}^-(p_x^+)$. Then Y also has only spacelike boundaries ($\hat{Y}$ is just Y with the two lines L^- and L^+ added back in) and is past-distinguishing; it is also generalized future-complete. Let $i : X \to Y$ just be inclusion; this clearly is continuous and generalized future-continuous and preserves spacelike boundaries (and it is a chronological isomorphism onto its image). Note that $\tilde{i}^g : \hat{X} \to Y$ takes each of the lines L^- and L^+ onto L. Since in $\hat{X}$, p_0^- is the limit of the curve L^-, as is p_0^+ for L^+, we'd better have both p_0^- and p_0^+ limits of the curve L in Y. And this is, indeed, the case: $\{p_0^-, p_0^+\}$ is a non-Hausdorff pair in Y, with any neighborhood of either point necessarily containing an end of the curve L, i.e., all p_x with $x < x_0$ for some $x_0 > 0$ (this can be seen by noting, for instance, that for any sequence $\sigma = \{p_{x_n}\}$ with x_n going to 0, $p_0^- \in L_Y(\sigma)$—since p_0^- is regular, just use the original definition of $\hat{\ }$-limit—so any closed set containing such a σ contains p_0^-, so any closed set excluding p_0^- excludes any such sequence, so any neighborhood of p_0^- contains an end of L). The $\hat{\ }$-topology on Y is precisely the same as the quotient topology on $\hat{X}/\sim$.

Now consider a case with the boundary being null: Let Π be the null plane $\{z = y\}$ in $\mathbb{L}^3$, and let $X = I^-(\Pi)$. Clearly $\hat{\partial}(X)$ can be identified with Π and $\hat{X}$ with the closure of X in $\mathbb{L}^3$. Do the same thing as above: For $x \geq 0$, let $p_x^- = (x,-1,-1)$ and $p_x^+ = (x,0,0)$; let $Y = \hat{X}/\sim$, where $\sim$ is the equivalence relation given by $p_x^- \sim p_x^+$ for each $x > 0$, and no other points identified together (let p_x name the corresponding equivalence class). With the half lines L^- and L^+ in $\hat{X}$ defined analogously as above, this equivalence relation identifies L^- and L^+ into a single half-line L. As before, define $I_Y^-(p_x) = I_{\hat{X}}^-(p_x^-) \cup I_{\hat{X}}^-(p_x^+)$. We have Y past-distinguishing and generalized future-complete. Let $i : X \to Y$ be inclusion: continuous and generalized future-continuous, and also a chronological isomorphism onto its image.

The interesting fact is that $\tilde{i}^g : \hat{X} \to Y$ is not continuous: p_0^- is not a limit of L (though p_0^+ is); more specifically, for any sequence σ lying in L with x-coordinate approaching 0, $p_0^- \notin L_Y(\sigma)$. The reason is that although every point in $I^-(p_0^-)$ is eventually in the past of σ, the same is also true for an IP which properly contains $I^-(p_0^-)$, namely, $I^-(p_0^+)$ (see Figure 10). The neighborhoods of p_0^+ are analogous to those in the previous example, but the neighborhoods of p_0^- are quite different: They include sets formed by starting with an open set in $\hat{X}$, and then deleting all points of L^- (or L^+). This gives Y some decidedly odd-looking open sets (though Y is still non-Hausdorff, as neighborhoods of p_0^- must contain points in a deleted neighborhood of an end of L^-, as must neighborhoods of p_0^+). The $\hat{\ }$-topology here is not at all the quotient topology on $\hat{X}$, but takes special cognizance of the null nature of the boundary and its relation to the identified points. (Note that $\hat{i} : \hat{X} \to \hat{Y}$ is continuous, even though X and Y are not in **Spbd**; $\hat{Y}$ consists of

Y with the curves L^- and L^+ added back in, not Hausdorff-separated from L; $\hat{i}$ takes L^- to L^-—not to L—and p_0^- is the limit of L^- in $\hat{Y}$. Theorem 3.6 is not applicable because Y has points—those in L—which are not future limits of chains in $i[X]$.)

The sense in which the boundary in this example is null is evident from the embedding into Minkowski space; but it can also be formally addressed in the manner of [GKP]: Starting with a set X with both a chronology relation $\ll$ and a causality relation $\prec$, obeying appropriate properties, one can extend $\prec$ (as well as $\ll$) to $\hat{X}$ by defining, for P and Q in $\hat{\partial}(X)$ and x in X, $x \prec P$ for $I^-(x) \subset P$, $P \prec x$ for $P \subset I^-(x)$, and $P \prec Q$ for $P \subsetneqq Q$. Then the boundary above possesses points $P \prec Q$, but with $P \ll Q$ failing.

Statement (3) of Proposition 4.6 looks rather like a universality principle, but it isn't quite, since we're not making use of the appropriate functor (generalized future completion). But it does mean that we have a strong characterization of any generalized future-completing, generalized past-distinguishing boundary on a chronological set with spacelike boundaries:

Proposition 4.7. *Let Y be a chronological set with only spacelike boundaries, pasts of points non-empty, generalized past-distinguishing, and generalized future-complete (i.e., in* **GFcplGPdisSpbdFtopGChron***). Then, in the $\hat{\ }$-topology, Y is a topological quotient of its future completion, $\hat{Y}$.*

Proof. Apply Proposition 4.6(3) to $1_Y : Y \to Y$, the identity map; this yields $\widetilde{1_Y}^g : \hat{Y} \to Y$, with $\widetilde{1_Y}^g \circ \hat{\imath}_Y = 1_Y$. Now make use of the categorical fact that if we have $\alpha : A \to B$ and $\beta : B \to A$ with $\beta \circ \alpha = 1_A$, then $A \cong B/\!\sim$, where $\sim$ is the equivalence relation defined by $b_1 \sim b_2$ if and only if $\beta(b_1) = \beta(b_2)$. (This has an easy categorical proof: For any $\phi : B \to C$ which respects the $\sim$ relation, there is a unique $\tilde{\phi} : B/\!\sim\, \to C$ with $\tilde{\phi} \circ \pi = \phi$, where $\pi : B \to B/\!\sim$ is the projection onto the quotient. Then apply this to β, obtaining $\tilde{\beta} : B/\!\sim\, \to A$ with $\tilde{\beta} \circ \pi = \beta$. We also have $\pi \circ \alpha : A \to B/\!\sim$. Then $\tilde{\beta} \circ \pi \circ \alpha = \beta \circ \alpha = 1_A$, and $\pi \circ \alpha \circ \tilde{\beta} = 1_{B/\sim}$, as can be directly calculated. Thus, $A \cong B/\!\sim$, with β the projection map.) Result: $Y \cong \hat{Y}/\!\sim$ with projection map $\widetilde{1_Y}^g$. $\square$

Proposition 4.7 identifies a generalized future-complete chronological set (in the spacelike boundaries category) as necessarily being derived, as a quotient, from a standard future completion; but it doesn't help us determine *which* future completion if we start from an incomplete object. What would be better would be an analogue of Theorem 3.6, giving a rigidity result for any general future-completing boundary, starting with a given chronological set. This, too, can be done, though spacelike boundaries must be assumed (the example above with a null boundary illustrates how things can go wrong for a generalized future completion, absent that assumption):

We'll model this the same way as in the proof of Theorem 3.6: Start with X, a generalized past-distinguishing chronological set with only spacelike boundaries and no points with empty pasts. To effect a generalized future completion of X, we need to embed X in a generalized future-complete chronological set, $i : X \to Y$ in **GPdisSpbdFtopGChron**; we require that Y be generalized future-complete, and that with $Y_0 = i[X]$ and $i_0 : X \to Y_0$ the restriction of i, that i_0 be a chronological isomorphism. We want Y to consist only of the image of X and of necessary

boundary points: So we also require that every point in $\partial(Y) = Y - Y_0$ be the generalized future limit of some future chain in Y_0. What we want is to show that Y is a topological quotient of $\hat{X}$ (not just of $\hat{Y}$) and also to identify $\partial(Y)$.

The proofs involved are perhaps the most intricate in this paper, but the results are among the most interesting; important applications arise in Theorem 5.3.

Theorem 4.8. *Let X be a generalized past-distinguishing chronological set with spacelike boundaries and no points with empty pasts. Let Y be any generalized past-distinguishing, generalized future completion of X (also with spacelike boundaries and no points with empty pasts); then, in the $\hat{\ }$-topology, Y is a topological quotient of $\hat{X}$.*

Proof. We have $i : X \to Y$ in **GPdisSpbdFtopGChron**, with Y generalized future-complete. Consequently, by Proposition 4.6(3), we have $\tilde{i}^g : \hat{X} \to Y$ with $\tilde{i}^g \circ \hat{\iota}_X = i$. (The continuity of this map is the only place where the spacelike boundaries play an essential role; but the example above, following Proposition 4.6, demonstrates the crucial aspect of that assumption.) We will establish a homeomorphism between Y and the quotient of $\hat{X}$ by the equivalence relationship $\sim$ defined by $p \sim q$ if and only if p and q have the same image under $\tilde{i}^g$. Let $\pi : \hat{X} \to \hat{X}/\!\sim$ denote the projection map. We automatically have a (unique) continuous map $j : \hat{X}/\!\sim \to Y$ such that $j \circ \pi = \tilde{i}^g$, and j is clearly injective. We need to see that j is onto and bicontinuous.

It will help to be clear about the action of the map $\tilde{i}^g$: For elements of X, $\tilde{i}^g$ is just the same as i. For $P \in \hat{\partial}(X)$, we look at any future chain c in X generating P and consider $i[c]$: As Y is generalized future-complete, this future chain must have a generalized future limit y; and as Y is generalized past-distinguishing, it has only one such. In other words, for some unique y, $P' = I^-[\,i[P]\,] \in \mathfrak{P}_y$; and $\tilde{i}^g(P) = y$. (Note that an image of an element of $\hat{\partial}(X)$ need not be in $\partial(Y)$: Suppose X has a non-regular point x; let P be a past component. Then P is in $\hat{\partial}(X)$—P cannot be $I^-(x')$, for then x' and x would share a past component, but X is assumed to be generalized past-distinguishing, and $x' \neq x$ since x is non-regular. Then $\tilde{i}^g(P) = i(x)$.)

Clearly, j is onto Y_0, since $j(\pi(x)) = i(x)$. To show j is also onto $\partial(Y)$, we need the fact that for any $z \in \partial(Y)$, there is a future chain c in X with z the generalized future limit of $i[c]$. Let $P = I^-[c]$. P must be in $\hat{\partial}(X)$: If $P = I^-(x)$, then with $P' = I^-[\,i[c]\,]$, we have $P' = I^-(i(x))$ (since $i_0 : X \to Y_0$ is a chronological isomorphism); but then P' is a common past component for $i(x)$ and z, so $i(x) = z$, so $z \in Y_0$. Knowing that $P \in \hat{\partial}(X)$, we have $\tilde{i}^g(P) = z$. Thus $\tilde{i}^g$ is onto Y, and j must be as well.

To show $j : \hat{X}/\!\sim \to Y$ is bicontinuous, we have to show that if a sequence in Y has a limit, then the same holds for the images under j^{-1}. By the same process used in Section 2 (before Proposition 2.1) to characterize the continuity of a map between topological spaces defined by limit-operators, we can characterize the current need this way: For any sequence σ and point y in Y, if $y \in L_Y(\sigma)$, then $j^{-1}(y)$ is in the closure of $j^{-1}[\sigma]$ (this uses the fact that a point is a limit of a sequence if and only if it is in the closure of all subsequences).

The quotient topology has, as open sets in $\hat{X}/\!\sim$, sets of the form $\pi[U]$ for $U \subset X$ open and full (i.e., if $p \in U$ and $q \sim p$, then $q \in U$ also); similarly, a closed set in $\hat{X}/\!\sim$ is anything of the form $\pi[A]$, where A is a closed and full subset of $\hat{X}$.

Thus, to show $\bar{p} \in \hat{X}/{\sim}$ in the closure of a sequence $\bar{\sigma}$, it suffices to show that for any subsequence $\bar{\tau} \subset \bar{\sigma}$, there is a subsubsequence $\bar{\rho} \subset \bar{\tau}$, a sequence ρ' in $\hat{X}$ with $\pi[\rho'] = \bar{\rho}$, and some point $p \in L_{\hat{X}}(\rho')$ with $\pi(p) = \bar{p}$—for then any closed and full set in $\hat{X}$ containing $\pi^{-1}[\bar{\sigma}]$ must contain $\pi^{-1}(\bar{p})$.

Thus, our objective comes down to the following (eschewing $\hat{X}/{\sim}$ altogether): to show that for any sequence σ in Y and $y \in L_Y(\sigma)$, for any subsequence $\tau \subset \sigma$, there is a sequence ρ' in $\hat{X}$ and $p \in L_{\hat{X}}(\rho')$ with $\tilde{i}^g[\rho'] \subset \tau$ and $\tilde{i}^g(p) = y$.

We will need a Lemma much like that in Proposition 2.4: For any IP P in Y_0, let $\bar{P} = I^-[P]$, where I^- denotes the past in Y (we'll use I_0^- for the past in Y_0); for any IP Q in Y, let $Q_0 = Q \cap Y_0$.

Lemma. *The maps $P \mapsto \bar{P}$ and $Q \mapsto Q_0$ establish an isomorphism of partially ordered sets (under inclusion) between $\mathcal{IP}(Y_0)$ and $\mathcal{IP}(Y)$.*

The proof of this Lemma is surprisingly complicated, though essentially technical; it relies on the assumption of spacelike boundaries, absent a separate hypothesis that Y_0 be chronologically dense in Y. The Lemma's proof will be delayed till after the other elements of the proof of this theorem.

Given a sequence σ in Y with $y \in L_Y(\sigma)$, consider any subsequence $\tau \subset \sigma$: We know there is a subsubsequence $\rho \subset \tau$ and a past component Q of y (in Y) with $Q \in \mathcal{L}_Y(\rho)$. Then Q_0 is generated by a future chain c_0 in Y_0, i.e., there is a chain c in X with $c_0 = i[c]$, so that $Q_0 = I_0^-[\,i[c]\,]$. Let $P = I^-[c]$, an IP in X. If for some $x \in X$, $P \in \mathfrak{P}_x$, then let $p = x$; otherwise, let $p = P$. In either case, we have P is a past component of p in $\hat{X}$ and $\tilde{i}^g(p) = y$: In the first case, x is a generalized future limit of c, so $i(x)$ must be a generalized future limit of $i[c] = c_0$ in Y_0; then $i(x)$ is also a generalized future limit of c_0 in Y (as is seen by applying the Lemma: Q_0 is maximal in $I_0^-(i(x))$ if and only if Q is maximal in $I^-(i(x))$), so, y being the unique generalized future limit of c_0 in Y, $i(x) = y$. In the second case, P *qua* subset of X is the only past component in $\hat{X}$ of P *qua* element of $\hat{X}$ (since X has spacelike boundaries, the past in $\hat{X}$ of P can contain no elements of $\hat{\partial}(X)$, for any such would then be properly contained by P; thus, $I^-_{\hat{X}}(P) = P$); since $i[c] = c_0$ generates, in Y, an IP which is a past component of y (namely, Q), we have $\tilde{i}^g(P) = y$.

So we have $P \in \mathfrak{P}_p$ where $\tilde{i}^g(p) = y$. What we need now is a sequence ρ' in $\hat{X}$ with $\tilde{i}^g[\rho'] = \rho$ and $P \in \mathcal{L}_{\hat{X}}(\rho')$ (for then $p \in L_{\hat{X}}(\rho')$). We will construct this now:

We know that $Q = I^-[\,i[c]\,] \in \mathcal{L}_Y(\rho)$; in other words, for each m, eventually $i(c(m)) \ll \rho(n)$: Specifically, for each m, there is some n_m such that for all $n > n_m$, $i(c(m)) \ll \rho(n)$; we may safely take the sequence of numbers $\{n_m\}$ to be monotonic increasing. Then whenever $n > n_m$, there is a past component Q_n^m of $\rho(n)$ containing $i(c(m))$. It is these past components in Y that will be used to generate the proper sequence in $\hat{X}$, but a diagonal process is needed to reduce from a doubly-indexed family to a singly-indexed one.

First we invert the roles of m and n above: For each n, let $M_n = \max\{m \mid n_m < n\}$; conceivably, some $M_n = \infty$ (and then all $M_{n'} = \infty$ for $n' > n$), though we don't expect that to be the case usually. As n increases, n eventually surpasses any one n_m, so if the $\{M_n\}$ are finite, they grow to infinity; thus, in any event, we can pick a monotonically increasing sequence $\{m_n\}$ such that $\{m_n\}$ goes to infinity and for all n, $m_n \leq M_n$; then for all n, for all $m \leq m_n$, $i(c(m)) \in Q_n^m$.

Now we diagonalize: For each n, let $Q_n = Q_n^{m_n}$; then for all n, for all $m \leq m_n$

$i(c(m)) \in Q_n \in \mathfrak{P}^Y_{\rho(n)}$. As before, for each n we can find a future chain c^n in X such that $Q_n = I^-[\,i[c^n]\,]$. We have for all $m \le m_n$, there is some j with $i(c(m)) \ll i(c^n(j))$; therefore, $c(m) \ll c^n(j)$. Thus, if we let $P_n = I^-[c^n]$, we have for all $m \le m_n$, $c(m) \in P_n$. These $\{P_n\}$ will be the basis for our sequence ρ': For each n, if there is some $x_n \in X$ with P_n a past component of x_n, then let $\rho'(n) = x_n$; otherwise, let $\rho'(n) = P_n$, an element of $\hat{\partial}(X)$. Note that in the first case, x_n is a generalized future limit of c^n, so $i(x^n)$ is the generalized future limit in Y_0 of $i[c^n]$, and the same is true in Y (the Lemma guaranteeing the preservation of relations among past components); therefore, $\rho(n)$ also being the generalized future limit of $i[c^n]$, we have $i(x_n) = \rho(n)$. In the second case, $\tilde{i}^g(P_n)$ is the generalized future limit of $i[c^n]$—again, $\rho(n)$. Thus, in either case, $\tilde{i}^g[\rho'] = \rho$.

To show $P \in \mathcal{L}_X(\rho')$, we will need to have, for each m, eventually $c(m) \in P_n$. What we have is that for each n, for all $m \le m_n$, $c(m) \in P_n$. Concentrate attention on any fixed m: Since the sequence $\{m_n\}$ is increasing to infinity, for all n sufficiently large, m will be less than m_n, placing $c(m)$ in P_n.

To check for maximality of P, suppose we have $P' \in \mathcal{IP}(\hat{X})$ with $P' \supset P$ and for all $q \in P'$, for some $\{n_k\}$, for all k, $q \ll \rho'(n_k)$. Actually, since X has only spacelike boundaries, for all $q \in P'$, q is in X, not in $\hat{\partial}(X)$, so we can write P' simply as $I^-_X[c']$ for some chain c' in X (though we could always use the Lemma in Theorem 2.4 to write any IP in any $\hat{X}$ as the $\hat{X}$-past of a chain in X). Let $Q' = I^-_0[\,i[c']\,]$; from $P' \supset P$, *i.e.*, $I^-[c'] \supset I^-[c]$ in X, we have $I^-_0[c'] \supset I^-_0[c]$ *i.e.*, $Q' \supset Q_0$, since Y_0 is just a reflection of X. Then we also have $\overline{Q'} = I^-[\,i[c']\,] \supset Q$.

We have for all n and k, $c'(n) \ll \rho'(n_k)$. In case $\rho'(n_k) \in X$, this gives us $i(c'(n)) \ll i(\rho'(n_k)) = \rho(n_k)$. In case $\rho'(n_k) \in \hat{\partial}(X)$, $\rho'(n_k) = I^-[c^{n_k}]$, and we have for all j sufficiently high, $c'(n) \ll c^{n_k}(j)$, whence $i(c'(n)) \ll i(c^{n_k}(j))$; thus, $i(c'(n)) \in I^-[\,i[c^{n_k}]\,] = Q_{n_k}$, so $i(c'(n)) \ll \rho(n_k)$. Therefore, we have that for all $q \in \overline{Q'}$, $q \ll \rho(n_k)$. It follows that $\overline{Q'} = Q$, whence $Q' = Q_0$, whence $P' = P$. Thus, $P \in \mathcal{L}(\rho')$. $\square$

Proof of Lemma. If $P \in \mathcal{IP}(Y_0)$ is generated by a chain c, then $\bar{P} = I^-[c]$, so $\bar{P} \in \mathcal{IP}(Y)$; thus $\overline{(\ )} : \mathcal{IP}(Y_0) \to \mathcal{IP}(Y)$. It takes a bit of doing to go the other way around: We must first establish that for any future chain c in Y, there is an interweaving chain c_0 in Y_0; this is where the spacelike character of the boundaries of Y comes into play. Once we know that, then for any $Q \in \mathcal{IP}(Y)$, we take a chain c generating Q and then an interweaving chain c_0 in Y_0; then Q is also generated by c_0, so $Q_0 = I^-[c_0] \cap Y_0 = I^-[c_0] \in \mathcal{IP}(Y)$ and $(\ )_0 : \mathcal{IP}(Y) \to \mathcal{IP}(Y_0)$. Then these maps are inverses of one another: For $P = I^-_0[c]$, we have $(\bar{P})_0 = (I^-[c])_0 = I^-[c] \cap Y_0 = P$ (since c is already in Y_0); and for $Q = I^-[c]$, with interweaving chain c_0 in Y_0, we have $\overline{Q_0} = I^-[c_0] = Q$. The preservation of inclusion by each map is clear.

So we need to establish the existence of interweaving chains: Given a chain $c = \{z_n\}$ in Y, we want to find a chain $c_0 = \{y_n\}$ in Y_0 with $z_{n-1} \ll y_n \ll z_n$. We might as well assume all z_n are in $\partial(Y)$: If only a finite number aren't we can just drop them, and if a subsequence lies in $\partial(Y)$, we can concentrate our attention on that subsequence. Then each z_n is the generalized future limit of a chain $c^n = \{y^n_m\}_{m \ge 1}$ in Y_0. For each n, let $P_n = I^-[c^n]$, and let Q_n be the past component of z_n containing z_{n-1}. We will show that for each n, P_n and Q_n must be the same; then, since Q_n is generated by c^n, there must be some m with

$z_{n-1} \ll y^n_m \ll z_n$, so letting $y_n = y^n_m$ finishes the job.

Fix n; we will show z_n must be regular (see Figure 11). Suppose z_n is non-regular; then P_n must be in $\hat{\partial}(Y)$: If $P_n = I^-(w)$ for some w, then, w and z_n share a past component (namely, P_n), so, by generalized past-distinguishing, $w = z_n$; but that can't be, because w is regular and z_n isn't. We have for all m, $y^n_m \ll z_n \ll z_{n+1}$; therefore, $P_n \subset Q_{n+1}$. But since all elements of $\hat{\partial}(Y)$ are inobservable, this implies $P_n = Q_{n+1}$. Therefore, since $z_n \in Q_{n+1}$, for some m, $z_n \ll y^n_m$; but we also know that for all m, $y^n_m \ll z_n$, so we have a contradiction. Ergo, z_n must be regular.

Once we know z_n is regular, all its past components are the same: $P_n = Q_n$, and we are done. □

As a result of Theorem 4.8, to identify any generalized future-completing boundary for X, we need only look in $\hat{X}$ and its quotients by equivalence relations (actually, quotients by continuous maps). However, although X sits nicely in $\hat{X}$, once we apply the equivalence relation to get $\hat{X}/\!\sim$, the image of X is no longer sharply separated from the boundary, since non-regular points of X get identified with their past components in the projection to $\hat{X}/\!\sim$. Thus, a cleaner picture is available in $\hat{X}^g$, which omits from consideration those IPs of X which are past components, so that nothing in $\hat{\partial}^g(X)$ becomes identified with a point of X under an analogous equivalence relation. This gives us a clean statement for $\partial(Y)$:

Corollary 4.9. *Let X be a generalized past-distinguishing chronological set with spacelike boundaries and no points with empty pasts. Let Y be any generalized past-distinguishing, generalized future completion of X (also with spacelike boundaries and no points with empty pasts); then $\partial(Y)$ (the generalized future-completing boundary) is a topological quotient of $\hat{\partial}^g(X)$, the Generalized Future Chronological Boundary of X.*

Proof. Since, by Theorem 4.8, Y is homeomorphic to $\hat{X}/\!\sim$ (for $\sim$ the equivalence relation defined by $\tilde{i}^g : \hat{X} \to Y$), we need concern ourselves only with identifying the boundary placed on X in $\hat{X}/\!\sim$. But, as related above, we really want, instead, to look at $\hat{X}^g/\!\sim$, where the $\sim$ relation is essentially the same: $\hat{X}^g$ is a subset of $\hat{X}$, just omitting those elements of $\hat{\partial}(X)$ which are past components of non-regular points in X. Since $\tilde{i}^g$ maps the past components of a non-regular point x to $i(x)$, all the equivalence classes in $\hat{X}$ have a representative in $\hat{X}^g$.

Also note that the chronological sets $\hat{X}^g$ and $\hat{X}$ bear the relation of X and $\bar{X}$ in Theorem 4.4, since no elements of $\hat{\partial}(X)$ are chronologically related; thus, $\hat{X}^g$, with the induced chronology relation, has the subspace topology from $\hat{X}$.

We will put this all together to show that $\hat{X}^g/\!\sim$ is homeomorphic to $\hat{X}/\!\sim$:

We have the situation of topological spaces $A \subset B$ with an equivalence relation $\sim$ on B, and the equivalence relation passing to the subspace A. With $\pi : B \to B/\!\sim$ being the projection to the quotient space and $j : A \to B$ the inclusion map, we have $\pi \circ j : A \to B/\!\sim$ is continuous. Since this map respects the induced equivalence relation on A, we have the continuous map $\tilde{j} : A/\!\sim \to B/\!\sim$, i.e., $\tilde{j} : \hat{X}^g/\!\sim \to \hat{X}/\!\sim$. Now we need to construct a continuous inverse for $\tilde{j}$:

Since $\hat{X}^g$ is generalized past-distinguishing and generalized future-complete, we can apply Proposition 4.6(3) to $\hat{\imath}^g_X : X \to \hat{X}^g$ (where $\hat{\imath}^g_X$ is the obvious inclusion), yielding a map $\widetilde{\hat{\imath}^g_X}^{\,g} : \hat{X} \to \hat{X}^g$; this is the map π_X forming the natural transformation π. To understand its action, it is best to think of $\hat{\partial}(X)$ as broken into two

parts: $\bigcup\{\mathfrak{P}_x \mid x \text{ is non-regular}\}$ and $\hat{\partial}^g(X)$. The action of π_X is to take $x \in X$ to x; for non-regular x, to take $P \in \mathfrak{P}_x$ to x (since x is then the generalized future limit of a chain generating P); and to take $P \in \hat{\partial}^g(X)$ to P (since P is then itself the future limit of its generating chain). Since the relation $\sim$ from Theorem 4.8 always identifies, for non-regular x, all elements of $\mathfrak{P}_x$ with x, we see that π_X respects $\sim$; thus, as above, π_X induces a map $\tilde{\pi}_X : \hat{X}/\!\sim \to \hat{X}^g/\!\sim$.

The map $\pi_X \circ j : \hat{X}^g \to \hat{X}^g$ is just the identity, so $\tilde{\pi}_X \circ \tilde{j} : \hat{X}^g/\!\sim \to \hat{X}^g/\!\sim$ is also the identity. The map $j \circ \pi_X : \hat{X} \to \hat{X}$ is the identity on $\hat{X}^g$; for the remainder—$P \in \mathfrak{P}_x$ for x non-regular—it takes P to x. However, note that $P \sim x$ in that situation (essentially because the map $\tilde{i}^g$ itself factors through π_X: $\tilde{i}^g = \hat{i}^g \circ \pi_X$, as alluded to in the proof of Proposition 4.6(3)); thus, we have $\widetilde{j \circ \pi_X} = \tilde{j} \circ \tilde{\pi}_X$ is the identity on $\hat{X}/\!\sim$. Therefore, we have a homeomorphism $\hat{X}^g/\!\sim \,\cong \hat{X}/\!\sim$.

Finally, we note that X is nicely embedded in $\hat{X}^g/\!\sim$: If $\tilde{i}^g(P) = x$ for $P \in \hat{\partial}(X)$, then $i(x)$ is a generalized future limit for the image of a chain generating P; but this reflects in X, implying that P is a past component of x. If x is regular, then $P = I^-(x)$, and P is not in $\hat{\partial}(X)$; and if x is non-regular, then P is not in $\hat{\partial}^g(X)$. Therefore, no element of X can be identified with any element of $\hat{\partial}^g(X)$ by $\sim$. Thus, the boundary attached to X by $\hat{X}^g/\!\sim$ (ergo, by Y) is just what is left over: $\hat{\partial}^g(X)/\!\sim$. $\square$

We close with a medley of categorical observations:

In virtue of statement (2) of Proposition 4.6, we have a categorical formulation of future completion, including universality, without the assumption of regularity, so long as we strengthen past-distinguishing to generalized past-distinguishing (**GPdis**):

Theorem 4.10. *Future completion is a functor* $\hat{}$: **GPdisSpbdFtopGChron** $\to$ **FcplGPdisSpbdFtopGChron**; *together with the natural transformation* $\hat{\boldsymbol{\iota}}$*, this forms a left adjoint to the forgetful functor.*

Proof. Proposition 4.6 provides the functoriality of future completion for these categories. All we need to observe is that the maps $\hat{\iota}_X : X \to \hat{X}$ are generalized future-continuous (obvious, since any generalized future limit in X is also one in $\hat{X}$) and $\hat{}$-continuous (Corollary 4.5). $\square$

We can also establish categorical results for past-determination without the regularity assumption. First we need an analogue of Proposition 3.2:

Proposition 4.11. *For any chronological set X with pasts of all points non-empty, $\iota^p_X : X \to X^p$ is a $\hat{}$-homeomorphism.*

Proof. Let the maps $P \mapsto P^p$ and $Q \mapsto Q_0$ be as in Lemma 3.1. Let L, $\mathcal{L}$, L^p, and $\mathcal{L}^p$ denote the various limit-operators in X and X^p; let $\mathfrak{P}_x$ and $\mathfrak{P}^p_x$ denote the families of past components of x in X and in X^p.

We will see that for any sequence σ, $L(\sigma) = L^p(\sigma)$.

Let σ be any sequence in X with $x \in L(\sigma)$. For any $\tau \subset \sigma$, there are $\rho \subset \tau$ and $P \in \mathfrak{P}_x$ with $P \in \mathcal{L}(\rho)$. Then $P^p \in \mathfrak{P}^p_x$; to have $x \in L^p(\sigma)$, we just need $P^p \in \mathcal{L}^p(\rho)$: For any $y \in P^p$, there is some $y_0 \in P$ with $y \ll^p y_0$, and $y_0 \ll \rho(n)$ (eventually) implies $y \ll^p \rho(n)$ (eventually). For any $Q \in \mathcal{IP}(X^p)$ with $Q \supset P^p$, suppose for all $y \in Q$, $y \ll^p \rho(n_k)$ (for some subsequence); then for all $y \in Q_0$, we can find $y' \in Q$ with $y \ll y'$, and $y' \ll \rho(n_k)$ implies $y \ll \rho(n_k)$. We also know

$Q_0 \supset P$, so $Q_0 = P$; therefore, $Q = P^p$. This finishes showing $P^p \in \mathcal{L}^p(\rho)$. Thus, $x \in L^p(\sigma)$.

A formally identical proof, with the roles of X and X^p reversed, establishes that $x \in L^p(\sigma)$ implies $x \in L(\sigma)$. With identical limit-operators, X and X^p have the same $\widehat{\ }$-topologies, with ι^p_X a homeomorphism. $\square$

With past determination well in hand, we obtain results for the GKP Future Causal Boundary construction, much as in Theorems 3.4 and 3.5:

Theorem 4.12.

(1) *Past determination is a functor* $\mathbf{p} : \mathbf{FtopGChron} \to \mathbf{PdetFtopGChron}$ *and also* $\mathbf{p} : \mathbf{SpbdFtopGChron} \to \mathbf{PdetSpbdFtopGChron}$*; together with the natural transformation* $\boldsymbol{\iota}^{\mathbf{p}}$*, these are left adjoints for the respective forgetful functors. In particular, for any map* $f : X \to Y$ *in* $\mathbf{FtopGChron}$ *and* Y *past-determined, there is a unique generalized future-continuous and* $\widehat{\ }$*-continuous map* $f^p : X^p \to Y$ *with* $f^p \circ \iota^p_X = f$*; and if* f *is also in* $\mathbf{Spbd}$*, so is* f^p*.*

(2) *There is a functor* $+ : \mathbf{GPdisSpbdFtopGChron} \to \mathbf{FcplPdetGPdis\text{-}SpbdFtopGchron}$ *and a natural transformation* $\boldsymbol{\iota}^+$*, forming a left adjoint to the forgetful functor.*

(3) *For any map* $f : X \to Y$ *in* $\mathbf{SpbdFtopGChron}$ *with* Y *generalized future-complete, generalized past-distinguishing, and past-determined, there is a unique generalized future-continuous and* $\widehat{\ }$*-continuous map* $\tilde{f}^{+g} : X^+ \to Y$ *with* $\tilde{f}^{+g} \circ \iota^+_X = f$*.*

Proof. (1) The same proof as in Theorem 3.3 applies for the functoriality of $\mathbf{p}$. The statements about f^p when Y is past-determined just amount to the universality property. (They are included here for comparison with statement (3) here and statement (3) of Proposition 4.6; these are more general than the analogous statements with Y being generalized past-determined.)

(2) The same proof applies as in Theorem 3.4.

(3) This follows the pattern of the universality properties, even though it is not categorical: With f in $\mathbf{SpbdFtopGChron}$ and Y also generalized future-complete and generalized past-distinguishing, Proposition 4.6(3) gives us $\tilde{f}^g : \hat{X} \to Y$ (unique with $\tilde{f}^g \circ \hat{\iota}_X = f$). Then with $\tilde{f}^g$ in $\mathbf{SpbdFtopGChron}$ and Y also past-determined, statement (1) above gives us $(\tilde{f}^g)^p : (\hat{X})^p \to Y$ (unique with $(\tilde{f}^g)^p \circ \iota^p_{\hat{X}} = \tilde{f}^g$). Then $\tilde{f}^{+g}$ is just $(\tilde{f}^g)^p \circ (j_X)^{-1}$ (unique for $\tilde{f}^{+g} \circ \iota^+_X = f$, where $\iota^+_X = j_X \circ \iota^p_{\hat{X}} \circ \hat{\iota}_X$). $\square$

5. Examples

This Section will largely be devoted to examining in detail the Future Chronological Boundary of a class of spacetimes with spacelike boundary, comparing the $\widehat{\ }$-topology with what might be expected; and to examining how the $\widehat{\ }$-topology works for boundaries derived in a simple manner from embedding spacetimes with spacelike boundaries into larger manifolds, where non-regular points come into play. (All the explicit spacetimes examined in this Section are globally hyperbolic, so there is no need to be concerned about past-determination: The $\widehat{\ }$ and $+$ functors are the same for these spaces, and what is I am calling the Future Chronological Boundary here could just as easily be called the GKP Future Causal Boundary.)

In a nutshell: The $\hat{\ }$-topology gives the "right" results in a variety of situations. However, it does give different results from the embedding topology for some embeddings which might be deemed questionable.

But we will begin with the most elementary of examples: Minkowski n-space, $\mathbb{L}^n$.

5.1 Minkowski Space.

The causal structure of Minkowski space is well-known, with explication in such sources as [HE]. But a small amount of detail here is not out of place. Let us split $\mathbb{L}^n$ orthogonally as $\mathbb{R}^{n-1} \times \mathbb{L}^1$; inside $\mathbb{R}^{n-1}$ we'll locate the unit sphere about the origin, $\mathbb{S}^{n-2}$. It is evident that any null line β in $\mathbb{L}^n$ gives rise to an IP in the form of $I^-[\beta]$, and that this is the same as the past of the null hyperplane determined by β: Specifically, if β is given by $\beta(s) = (z, 0) + s(p, 1)$ for $z \in \mathbb{R}^{n-1}$ and $p \in \mathbb{S}^{n-2}$, the null hyperplane bounding the corresponding IP is $\Pi = \{(x, t) \mid t = \langle x, p\rangle - \langle z, p\rangle\}$ (where $\langle\, ,\, \rangle$ denotes the Euclidean inner product in $\mathbb{R}^{n-1}$). Thus, the collection of IPs definable this way corresponds to the set of all null hyperplanes, parametrized by $\mathbb{S}^{n-2} \times \mathbb{R}^1$ as $\Pi_{p,a} = \{(x, t) \mid t = \langle x, p\rangle + a\}$; let $P_{p,a} = I^-[\Pi_{p,a}] = \{(x, t) \mid t < \langle x, p\rangle + a\}$. (These are the IPs constituting $\mathfrak{I}^+$, future null infinity.)

In fact, these are all the IPs of $\mathbb{L}^n$, aside from those which are pasts of a single point and the IP P_∞, often known as i^+ (or future timelike infinity), which consists of the entire spacetime. Sketch of a proof: Let γ be any future-endless timelike curve; we can represent γ by $\gamma(s) = (c(s), s)$ for $c : \mathbb{R} \to \mathbb{R}^{n-1}$ a curve with $|\dot{c}| < 1$ (where $|\ |$ denotes Euclidean length). It's not hard to see that $I^-[\gamma] = \{(x, t) \mid t < \sup_s(s - |c(s) - x|)\}$, i.e., the past of the graph of the function $b : x \mapsto \sup_s(s - |c(s) - x|)$. For any $x \in \mathbb{R}^{n-1}$, let $b_x : \mathbb{R} \to \mathbb{R}$ be the function given by $b_x(s) = s - |c(s) - x|$; this is monotonic increasing, so $b(x) = \sup b_x = \lim_{s\to\infty} b_x(s)$. Since for all s, $x \mapsto b_x(s)$ is Lipschitz-1, either for all $x \in \mathbb{R}^{n-1}$, $b(x) = \infty$; or for all x, $b(x)$ is finite, and b is Lipschitz-1. In the first case, $I^-[\gamma] = P_\infty$; in the second case, $I^-[\gamma] = P_{p,a}$ where $p = \lim_{s\to\infty} c(s)/|c(s)|$ and $a = b(0)$ (the fact that for all x, b_x has a limit, is what shows that $c(s)/|c(s)|$ has a limit; then calculation shows $b(x) = \langle x, p\rangle + a$).

Thus we know $\hat{\partial}(\mathbb{L}^n) = \{P_{p,a} \mid (p, a) \in \mathbb{S}^{n-2} \times \mathbb{R}^1\} \cup \{P_\infty\}$, giving us a nice parametrization of the boundary as a cone over the $(n-2)$-sphere. Indeed, that is the topology of the future boundary of $\mathbb{L}^n$ in its conformal embedding into the Einstein static spacetime, $\mathbb{S}^{n-1} \times \mathbb{L}^1$: This embedding can be described as follows (taken from [HE], Section 5.1) : Express $\mathbb{R}^{n-1}$ as $\mathbb{S}^{n-2} \times \mathbb{R}^+$ via $x \mapsto (x/|x|, |x|)$, so that $\mathbb{L}^n = \mathbb{S}^{n-2} \times \mathbb{R}^+ \times \mathbb{L}^1$; express $\mathbb{S}^{n-1}$ as a subset of $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}^1$, so that $\mathbb{S}^{n-1} \times \mathbb{L}^1$ is a subset of $\mathbb{R}^{n-1} \times \mathbb{R}^1 \times \mathbb{L}^1$. Then we define the map $\phi : \mathbb{S}^{n-2} \times \mathbb{R}^+ \times \mathbb{L}^1 \to \mathbb{R}^{n-1} \times \mathbb{R}^1 \times \mathbb{L}^1$ by

$$\phi(p, r, t) = (p \sin\theta, \cos\theta, \tau),$$

where

$$\theta = \tan^{-1}(t + r) - \tan^{-1}(t - r)$$
$$\text{and } \tau = \tan^{-1}(t + r) + \tan^{-1}(t - r).$$

Since this is a conformal map, the causal structure is preserved. For a fixed value of τ with $0 < \tau < \pi$, the image of ϕ has $|\theta| < \pi - \tau$. This yields, for the boundary of

the image of ϕ in the $\tau > 0$ region, a copy of $\mathbb{S}^{n-2}$ as $(\mathbb{S}^{n-2}\sin(\pi-\tau), \cos(\pi-\tau), \tau)$. Thus, the future boundary of the embedding (i.e., with $0 < \tau < \pi$) is $\mathbb{S}^{n-2} \times (0, \pi)$ together with the boundary-element at $\tau = \pi$, the point $(\mathbf{0}, 1, \pi)$: in other words, a cone on $\mathbb{S}^{n-2}$.

To explore the $\hat{}$-topology of $\hat{\partial}(\mathbb{L}^n)$, we need to know when it is that one IP can contain another. This is quite simple: Every $P_{p,a}$ is contained in P_∞, and $P_{p,a} \subset P_{q,b}$ if and only if $p = q$ and $a \leq b$ (any two hyperplanes will intersect if they aren't exactly parallel). For questions of convergence, it's easiest to work with a generating chain for each IP: $P_{p,a}$ is generated by $c_{p,a}(m) = (mp, m + a - 1/m)$. Then a sequence of IPs $\sigma = \{P_{p_n,a_n}\}$ approaches an IP $P_{p,a}$ (i.e.,$P_{p,a} \in L(\sigma)$) if and only if

(1) for each m, eventually $c_{p,a}(m) \in \sigma(n)$, and
(2) for any $\epsilon > 0$, for all m sufficiently large, eventually $c_{p,a+\epsilon}(m) \notin \sigma(n)$.

(The second clause is the contrapositive of saying that any IP containing $P_{p,a}$ and having each element of its generating chain eventually in the past of σ, must actually be $P_{p,a}$.) It is easy to work this out and show it equivalent to $\lim_{n\to\infty}(p_n, a_n) = (p, a)$ in the usual topology of $\mathbb{S}^{n-2} \times \mathbb{R}^1$. (For example: If for some sequence $\{n_k\}$, $\langle p, p_{n_k}\rangle < 1 - \delta$, then by clause (1), for all m, eventually $m + a - 1/m < m\langle p, p_{n_k}\rangle + a_{n_k}$, so eventually $a_{n_k} > m\delta + a - 1/m$. This means $\lim a_{n_k} = \infty$; but that contradicts clause (2).) It is also easy to show σ approaches P_∞ if and only if $\lim a_n = \infty$. Thus, the $\hat{}$-topology for the boundary is precisely that of a cone on $\mathbb{S}^{n-2}$, just as in the conformal embedding into $\mathbb{S}^{n-1} \times \mathbb{L}^1$.

We also need to be concerned about how the boundary is topologically connected to the spacetime: When is it that σ has $P_{p,a}$ (or P_∞) as a limit for $\sigma(n) = (x_n, t_n) \in \mathbb{L}^n$? This is considerably messier to work out, and the answer is perhaps a bit surprising: $P_{p,a} \in L(\sigma)$ if and only if

(1) $\lim t_n = \infty$,
(2) $\lim x_n/|x_n| = p$, and
(3) $\lim(t_n - |x_n|) = a$.

One might have expected condition (3) to be, instead, $\lim(t_n - \langle x_n, p\rangle) = a$, in light of the defining equation for $P_{p,a}$; but this turns out not to be a sufficient condition. However, the condition as given is precisely that needed for $\phi[\sigma]$ to have as limit the point $(p\sin(\pi-\tau), \cos(\pi-\tau), \tau)$ in the boundary of the image of ϕ, where $\tau = \tan^{-1} a$.

Just a word on how to establish this result: The two clauses needed for convergence to $P_{p,a}$ are

(1) for all m, eventually, $t_n - |x_n - mp| - m > a - 1/m$, and
(2) for any $\epsilon > 0$, for m sufficiently large, eventually $t_n - |x_n - mp| - m \leq a + \epsilon - 1/m$.

The key is to replace the expression on the left of the inequalities by one that is independent of m, that being $t_n - |x_n|$. The difference, $\delta_n^m = |x_n| - |x_n - mp| - m$, is always non-positive, but must be shown to go to 0 for fixed m. This turns out to be the case so long as $\{\lambda_n\}$ goes to infinity and $\{\mu_n/\lambda_n\}$ goes to 0, where $x_n = \lambda_n p + \mu_n q_n$ for q_n unit-length and perpendicular to p.

It is easy to show σ approaches P_∞ if and only if $\lim a_n = \infty$.

This is all summarized thus:

Proposition 5.1. *Let $M = \mathbb{L}^n$ and $E = \mathbb{S}^{n-1} \times \mathbb{L}^1$, and let $\phi : M \to E$ be the*

standard conformal embedding of Minkowski space in the Einstein static spacetime. Then $\hat{\phi} : \hat{M} \to \hat{E}$ is a homeomorphism onto its image, with the $\hat{}$-topology for $\hat{M}$; in particular, $\hat{\partial}(M)$ is a cone on $\mathbb{S}^{n-2}$. $\square$

It is worth noting the contrast between this result and those in Proposition 2.7 and in Theorem 3.6: Theorem 3.6 states that any future-completing boundary on M must be homeomorphic to $\hat{\partial}(M)$—but that assumes the use of the $\hat{}$-topology for the boundary, and the point of Proposition 5.1 is to use a boundary coming from an embedding in another spacetime. Proposition 2.7 fares better in this regard, in respect of the continuity of $\hat{\phi}$, as in this case, the image of $\hat{\phi}$ lies entirely within E, so that there is no concern over which topology to use for $\hat{\partial}(E)$—but that Proposition applies only in the case of spacelike boundaries, and M has a null boundary.

It is also worth contrasting the $\hat{}$-topology for $\hat{\partial}(\mathbb{L}^n)$ with that given in the original GKP topology for the Causal Boundary. This is explicated in [HE], Section 6.8: $M^\#$ is defined to be $\hat{M} \cup \check{M}$ (where $\check{M}$ is M plus the Past Chronological Boundary, formed of IFs). For any IF F in M, the sets F^{int} and F^{ext} are considered to be open sets in $M^\#$, where $F^{\text{int}} = F \cup \{P \in \hat{\partial}(M) \mid P \cap F \neq \emptyset\}$ and $F^{\text{ext}} = (M - \mathit{closure}(F)) \cup \{P \in \hat{\partial}(M) \mid \text{for any } A \text{ with } P = I^-[A], I^+[A] \not\subset F\}$; and dually for P^{int} and P^{ext} for any IP P. The collection of all of these four types of sets provides a sub-basis for the GKP topology in $M^\#$.

Apply this to $M = \mathbb{L}^n$, examining $(F_{p,a})^{\text{ext}}$ for any $(p, a) \in \mathbb{S}^{n-2} \times \mathbb{R}^1$: The portion in $\mathbb{L}^n$ is unexceptional, making up $P_{p,a}$. But for which IPs $P_{q,b}$ (or P_∞) is it true that when expressed as $I^-[A]$, the A must have points in its future not in $F_{p,a}$? The only possibilities are those with $q = p$ and $b < a$, as anything else will intersect $F_{p,a}$ and will clearly be expressible as $I^-[A]$ for some $A \subset F_{p,a}$ (more precisely: $P_{p,a}$ is ruled out because it is expressible as $I^-[l]$ for l a null half-line lying in $\Pi_{p,a}$, the boundary of $F_{p,a}$, and $I^+[l] \subset F_{p,a}$). Thus, the GKP topology on $\mathbb{L}^n \cup \hat{\partial}(\mathbb{L}^n) \cup \check{\partial}(\mathbb{L}^n)$ has, as an open set, anything of the form $P_{p,a} \cup \{P_{p,b} \mid b < a\}$ (where $P_{p,a}$ is to be read as a subset of $\mathbb{L}^n$). Then the GKP topology on $\hat{\partial}(\mathbb{L}^n)$ has these as open sets: for any $(p, a) \in \mathbb{S}^{n-2} \times \mathbb{R}^1$, $\{P_{p,b} \mid b < a\}$. This is not at all the topology of a cone on $\mathbb{S}^{n-2}$. (The other GKP-open sets in $\hat{\partial}(\mathbb{L}^n)$ come from $(I^+((x,t)))^{\text{ext}}$, yielding $\{P_{p,b} \mid \langle x, p\rangle + b < t\}$—but these add nothing new, as they wholly contain the open sets already mentioned.) This topology would have $\{P_{p_n,a_n}\}$ approach $P_{p,a}$ if and only if all $p_n = p$ and those elements of $\{a_n\}$ which are greater than a, if infinite in number, approach a in the usual sense; and every sequence $\{P_{p_n,a_n}\}$ approaches P_∞ (since the only neighborhood of P_∞ is all of $M^\#$).

(The GKP construction of the Causal Boundary includes identifications on $M^\#$ to produce a Hausdorff M^*, but this has no bearing on $\mathbb{L}^n$, as no identifications are made. Other methods of identification within $M^\#$ have been proposed, such as by Szabados in [S] and by Budic and Sachs in [BS]; but these also make no identifications for $M = \mathbb{L}^n$.)

It is possible to generalize the techniques employed for $\mathbb{L}^n$ to any standard static spacetime. The results for topology of the Future Chronological Boundary are what one would hope for, though the causal structure can show some curious anomalies. But it is sufficiently complicated as to warrant appearance in a separate paper.

5.2 Multiply Warped Products.

Perhaps the next most obvious example to look at is Schwarzschild space, as

pecially interior Schwarzschild, inside the event horizon, as the boundary—the Schwarzschild singularity—is spacelike. Using r and t as the standard Schwarzschild coordinates (see, e.g, [HE], Section 5.5), which switch timelike/spacelike roles inside the event horizon, we have, as the metric g for the $r < 2m$ region,

$$g \ = \ -\frac{1}{\frac{2m}{r} - 1}(dr)^2 \ + \ (\frac{2m}{r} - 1)(dt)^2 \ + \ r^2 h_{\mathbb{S}^2},$$

where m is the mass and $h_{\mathbb{S}^2}$ is the standard metric on the unit 2-sphere. We can get all the causal information we need by looking at a conformal metric,

$$\bar{g} \ = \ -(dr)^2 \ + \ (\frac{2m}{r} - 1)^2(dt)^2 \ + \ r^2(\frac{2m}{r} - 1)h_{\mathbb{S}^2}.$$

In other words, interior Schwarzschild is conformal to the spacetime $(0, 2m) \times \mathbb{R}^1 \times \mathbb{S}^2$ with metric $-(dr)^2 + f_1(r)h_{\mathbb{R}^1} + f_2(r)h_{\mathbb{S}^2}$ (where $h_{\mathbb{R}^1}$ is the standard Riemannian metric on $\mathbb{R}^1$) for some specific functions f_1 and f_2. The natural expectation is that the Schwarzschild singularity—the Future Chronological Boundary of internal Schwarzschild—should be $\mathbb{R}^1 \times \mathbb{S}^2$; and we will see that this is precisely so, in the $\hat{}$-topology. But rather than look just at interior Schwarzschild, we will examine a large class of spacetimes of this same form. Let us call a spacetime a *multiply warped product spacetime* if it has the following form: The manifold is $M = (a, b) \times K_1 \times \cdots \times K_m$ for some $a < b$ (possibly infinite) and for some manifolds K_1 through K_m. Each K_i has a Riemannian metric h_i, and for each i there is a positive function $f_i : (a, b) \to \mathbb{R}^+$. The spacetime metric is $g = -(dt)^2 + f_1(t)h_1 + \cdots + f_m(t)h_m$ (more precisely: $g = -(dt)^2 + \sum_i (f_i \circ t)\pi_i^* h_i$, where $t : M \to (a, b)$ and $\pi_i : M \to K_i$ are projection). The Riemannian manifolds (K_i, h_i) will be called the spacelike factors, the functions f_i the warping functions.

Interior Schwarzschild (a spherically symmetric vacuum spacetime) is conformal to a multiply warped product spacetime with two spacelike factors, the standard $\mathbb{R}^1$ and $\mathbb{S}^2$. Robertson-Walker spacetimes (homogeneous and isotropic, perfect fluid; see, e.g, [HE], Section 5.3) are multiply warped product spacetimes, each with a single spacelike factor: A Robertson-Walker spacetime is $((\alpha, \omega) \times K, -(dt)^2 + r(t)^2 h)$, where (α, ω) is some interval in $\mathbb{R}^1$ (finite, infinite, or half-infinite), (K, h) is a constant-curvature Riemannian manifold, and $r : \mathbb{R}^1 \to \mathbb{R}^+$ is some positive function, scaling the size of the universe. The Kasner spacetimes (spatially homogenous and vacuum; see, e.g, [W], Section 7.2) are multiply warped product spacetimes with three spacelike factors: These have the form of $((0, \infty) \times \mathbb{R}^3, -(dt)^2 + t^{2p_1}(dx)^2 + t^{2p_2}(dy)^2 + t^{2p_3}(dz)^2)$ for some constants p_1, p_2, and p_3 satisfying $\sum_i p_i = \sum (p_i)^2 = 1$; this has three spacelike factors of standard $\mathbb{R}^1$.

Not all multiply warped product spacetimes have spacelike boundaries: If any of the spacelike factors is incomplete as a Riemannian manifold, then the Future Chronological Boundary has timelike regions (i.e., chronology relations between boundary points); and if any of the warping functions goes to 0 too quickly (for a finite future end of the interval (a, b)) or to infinity too slowly (for an infinite future end), then the Future Chronological Boundary has null or timelike relations (i.e., inclusion of some boundary point in another). But here we will consider only the spacelike case.

The matter addressed in the following proposition is not akin to that of Proposition 2.7, concerning the continuity of $\hat{f} : \hat{X} \to \hat{Y}$ (say, with a multiply warped product spacetime for X, $(a,b] \times K$ for $Y = \hat{Y}$, and inclusion for f), because in that proposition it is the $\hat{\ }$-topology that is used for Y, and here it is the product topology on $(a,b] \times K$ that is of interest. In fact, we will see how to put a chronology relation on $(a,b] \times K$, and the point of the proposition is that the $\hat{\ }$-topology from that is the same as the product topology. In similar vein, Theorem 3.6—concerned with any future-completing and past-distinguishing boundary—addresses a different concern than this proposition, as it, also, looks at the $\hat{\ }$-topology of such a boundary, and we are concerned now with a "naturally occurring" topology for a boundary.

Proposition 5.2. *Let M be conformal to a multiply warped product spacetime with timelike factor (a,b) (assume b to be the future-end of the interval), spacelike factors (K_i, h_i), and warping functions $f_i : (a,b) \to \mathbb{R}^+$ $(1 \leq i \leq m)$. Suppose that for each i,*

(1) *h_i is a complete Riemannian metric, and*
(2) *for some finite $c \in (a,b)$, $\int_c^b f_i^{-\frac{1}{2}} < \infty$.*

Then M has only spacelike boundaries; $\hat{M}$ is $\hat{\ }$-homeomorphic to $(a,b] \times K$, where $K = K_1 \times \cdots \times K_m$; and $\hat{\partial}(M)$ is $\hat{\ }$-homeomorphic to K, included in $\hat{M}$ as $\{b\} \times K$.

Proof. The proof is somewhat lengthy, though largely uncomplicated: First we must identify all the IPs in M; then we have to show $\hat{\partial}(M)$ is spacelike; and finally we have to show the $\hat{\ }$-topology reflects the topology of $(a,b] \times K$.

Step (1): Identifying $\hat{\partial}(M)$.

First we will construct a new chronological set (which will turn out, essentially, to be $\hat{M}$): Let the set be $\bar{M} = (a,b] \times K$, and extend the chronology relation $\ll$ from M to $\bar{M}$ by defining, for any t with $a < t < b$ and any x and y in K, $(t,x) \ll (b,y)$ if and only if there is a timelike curve $\gamma : [t,b) \to M$ with $\gamma(t) = (t,x)$ and $\lim_{s \to b} \gamma(s) = (b,y)$. Since γ can always be expressed as $\gamma(s) = (s, c(s))$ for some curve c in K, this is equivalent to there being a curve $c : [t,b) \to K$ with $c(t) = x$, $\lim_{s \to b} c(s) = y$, and for all s, $\sum_i f_i(s)^{\frac{1}{2}} |\dot{c}_i(s)|_i < 1$, where $c_i = \pi_i \circ c$ (π_i being projection to the ith spacelike factor) and $|\ |_i$ denotes norm with respect to h_i. No other chronology relations (aside from those in M) are defined in $\bar{M}$; it should be evident that $(\bar{M}, \ll)$ is a chronological set. We will use $\bar{I}^-$ and $\bar{I}^+$ to denote past and future within $\bar{M}$, while I^- and I^+ denote the same in M.

The crucial step is to note that, just as in a spacetime, $\bar{I}^+$ yields an open set in $\bar{M}$ (using the product topology for $(a,b] \times K$):

Lemma. *For any $(t,x) \in M$, $\bar{I}^+((t,x))$ is open in $\bar{M}$.*

Proof of Lemma. All we need show is that $\{y \in K \mid (t,x) \ll (b,y)\}$ is open in K, i.e., that for $(b,y) \gg (t,x)$, we can find a neighborhood U of y in K such that for all $z \in U$, $(b,z) \gg (t,x)$. Since we are dealing only with small neighborhoods of y, the differences in the metrics h_i become irrelevant, and the question reduces to a Euclidean one: Given continuous curves $c_i : [t,b] \to \mathbb{R}^{k_i}$, C^1 on $[t,b)$ with $\sum_i f_i(s)^{\frac{1}{2}} \|\dot{c}_i(s)\| < 1$ for $s < b$ ($\|\ \|$ denoting Euclidean norm), is there a neighborhood U of $c(b)$ in $\mathbb{R}^n$ ($n = \sum_i k_i$ and $c = (c_1, \ldots, c_m)$) such that for all $\bar{y} \in U$, there

are continuous curves $\bar{c}_i : [t,b] \to \mathbb{R}^{k_i}$, C^1 on $[t,b)$ with $\sum_i f_i(s)^{\frac{1}{2}} \|\dot{\bar{c}}_i(s)\| < 1$ for $s < b$, satisfying $\bar{c}_i(t) = c_i(t)$ and $\bar{c}_i(b) = \bar{y}_i$?

We need only consider each factor separately, i.e., deal with $m = 1$: Given continuous $c : [t,b] \to \mathbb{R}^k$, C^1 on $[t,b)$ with $f(s)^{\frac{1}{2}} \|\dot{c}(s)\| < 1$ for $s < b$, we need to find a family of variations $\bar{c}$ of c, starting at the same point and satisfying the same differential inequality, with endpoints at b forming a neighborhood of $c(b)$. Let $\epsilon(s) = f(s)^{-\frac{1}{2}} - \|\dot{c}(s)\|$; ϵ is positive and continuous on $[t,b)$. For any continuous $\theta : [t,b] \to \mathbb{R}^k$, C^1 on $[t,b)$ with $\|\dot{\theta}(s)\| < \epsilon(s)$ for $s < b$ and obeying $\theta(t) = 0$, the curve $\bar{c} = c + \theta$ is an acceptable variation of c. If we let Θ be the set of all such θ, then it is clear that $\{\theta(b) \mid \theta \in \Theta\}$ contains a neighborhood of 0 in $\mathbb{R}^k$ (just consider separately each one-dimensional projection). $\square$

For any $x \in K$, define $Q_x = \bar{I}^-((b,x))$. We will show that $\hat{\partial}(M) = \{Q_x \mid x \in K\}$.

First it is clear that any Q_x is an IP in M, as $Q_x = I^-[\gamma_x]$, where γ_x is the timelike curve given by $\gamma_x(s) = (s,x)$ for all s: Surely anything in the past of γ_x lies in Q_x. Conversely, for any $(t,y) \in Q_x$, i.e., $(t,y) \ll (b,x)$, since $\bar{I}^+((t,y))$ is an open neighborhood of (b,x) in $\bar{M}$, $\bar{I}^+((t,y))$ contains some (s,x) for $s < b$, i.e., $(t,y) \ll (s,x)$; thus, $(t,y) \in I^-[\gamma_x]$. Furthermore, $Q_x \in \hat{\partial}(M)$, as it is clear that any $I^-((t,y))$ contains no (s,z) with $s \geq t$, while Q_x contains (s,x) for s arbitrarily close to b.

Next, we show that every IP in $\hat{\partial}(M)$ is a Q_x for some $x \in K$: Let $P = I^-[\gamma]$ for γ a future-endless timelike curve in M, i.e., $\gamma(s) = (s, c(s))$ for $c : [t_0, b) \to K$ C^1 and satisfying $\sum_i f_i(s)^{\frac{1}{2}} |\dot{c}_i(s)|_i < 1$. In particular, for each i, $|\dot{c}_i(s)|_i < f_i(s)^{-\frac{1}{2}}$. Let L_i denote the Riemannian length functional in (K_i, h_i); then for each i, $L_i(c_i) = \int_{t_0}^b |\dot{c}_i(s)|\, ds < \int_{t_0}^b f_i(s)^{-\frac{1}{2}}\, ds$, which is finite. Therefore, since K_i is complete, c_i must have an endpoint $x_i = \lim_{s \to b} c_i(s)$. Let $x = (x_1, ..., x_m)$. Then $P = Q_x$:

Since c approaches x, γ approaches (b,x); thus, for any $(t,y) \in I^-[\gamma]$, there is a timelike curve from (t,y) to (b,x): $P \subset Q_x$. For any $(t,y) \in \bar{I}^-((b,x))$, we have $(b,x) \in \bar{I}^+((t,y))$, which is an open neighborhood in $\bar{M}$ of (b,x). Since γ approaches (b,x), eventually it enters $\bar{I}^+((t,y))$, so $(t,y) \in I^-[\gamma]$: $Q_x \subset P$.

Finally, we need to know that the $\{Q_x \mid x \in K\}$ are all distinct: Given $x \in K$, for each $j < m$, let Q_x^j be the intersection of Q_x with $N_x^j = \{(s,z) \mid z_i = x_i \text{ for all } i \neq j\}$. Let M_j be the warped product spacetime $((a,b) \times K_j,\ -(dt)^2 + f_j(t) h_j)$. Then $Q_x^j = \{(s, x_1, ..., x_{j-1}, y, x_{j+1}, ..., x_m) \mid (s,y) \ll (b, x_j) \text{ in } \bar{M}_j\}$ (the idea being that if there is a timelike curve in M from (t,z) to (b,x) and $(t,z) \in N_x^j$, then there is a timelike curve wholly within N_x^j between those points).

Now, in $\bar{M}_j$, we can identify $\bar{I}^-((b,x_j))$ as $\{(s,y) \mid d_j(y, x_j) < \int_s^b f_j^{-\frac{1}{2}}\}$, where d_j is the Riemannian distance function in K_j. For s sufficiently close to b, $\int_s^b f_j^{-\frac{1}{2}} <$ diam(K_j), where diam denotes diameter (in case this is not infinite). Therefore, if we let S_s denote the s-slice of $\bar{I}^-((b,x_j))$, i.e., $S_s = \bar{I}^-((b,x_j)) \cap (\{s\} \times K_j)$, then S_s changes, depending on what x_j is—in fact, S_s narrows down to x_j as s approaches b. Then we can express Q_x^j as $\{(s, x_1, ..., x_{j-1}, y, x_{j+1}, ..., x_m) \mid (s,y) \in P_j\}$, where $P_j = \bar{I}^-((b,x_j))$ is a region in M_j that uniquely identifies x_j.

Suppose $Q_x = Q_{x'}$; then for each j, $Q_x^j = Q_{x'}^j$, so P_j and P'_j must be the same region in M_j; thus, $x_j = x'_j$ for each j. Therefore, $x = x'$.

Thus, we have fully identified $\hat{\partial}(M)$ with $\{b\} \times K$ (and, hence, $\hat{M}$ with $(a,b] \times K$).

Step (2): $\hat{\partial}(M)$ is spacelike.

This is now quite easy: If $Q_x \subset Q_{x'}$, then, for each j, use the same s-slices through $\bar{I}^-((b, x_j))$ as above, yielding for all $s < b$, $S_s \subset S'_s$. Let $B_r(p)$ denote the ball of radius r around p in K_j in the Riemannian distance function; then $B_{r(s)}(x_j) \subset B_{r(s)}(x'_j)$, where $r(s) = \int_s^b f_j^{-\frac{1}{2}}$. Since $r(s)$ goes to 0 as s goes to b, if $x_j \neq x'_j$, we can find $s < b$ with $r(s) < d(x_j, x'_j)$, yielding a contradiction. Thus, for all j, $x_j = x'_j$, and $x = x'$.

Clearly, we cannot have $Q_x \subset I^-(t, y)$, so all elements of $\hat{\partial}(M)$ are inobservable. Since, by Proposition 2.6, $\hat{\partial}(M)$ is necessarily closed in $\hat{M}$ (i.e., in the $\hat{}$-topology; we have yet to verify that is the same as the product topology on $\bar{M}$), this is all we need to have M in the **Spbd** category.

Step (3): The topology on $\hat{M}$.

First consider a sequence σ lying in $\hat{\partial}(M)$, so $\sigma(n) = Q_{x_n}$ for some $x_n \in K$. We want to show that for any $y \in K$, $Q_x \in L(\sigma)$ if and only if $\lim x_n = y$. Since Q_x is inobservable, $Q_x \in L(\sigma)$ if and only if for all (t, y) with $t < b$, eventually $(t, y) \ll (b, x_n)$. If $\{x_n\}$ approaches y, then this is clearly true: $\bar{I}^+((t, y))$ is an open neighborhood of (b, y), so eventually $(b, x_n) \in \bar{I}^+((t, y))$. Conversely, suppose for all $t < b$, eventually $(t, y) \ll (b, x_n)$. Then, for each j, looking at the region in M_j as above, we have, eventually, $d_j(y_j, (x_n)_j) < \int_t^b f_j^{-\frac{1}{2}}$. For any integer k, we can find a $t_k < b$ such that for all j, $\int_{t_k}^b f_j^{-\frac{1}{2}} < 1/k$. Thus, for each j, for any k, eventually $d_j(y_j, (x_n)_j) < 1/k$: $\{(x_n)_j\}$ must approach y_j for each j, so $\{x_n\}$ approaches y.

Since $\hat{\partial}(M)$ is closed in $\hat{M}$ (M being a spacetime), we don't have to consider the possibility of a sequence of boundary elements approaching a point of M.

The last thing to consider is a sequence in M, $\sigma(n) = (t_n, x_n)$; we need to show that for any $y \in K$, $Q_y \in L(\sigma)$ if and only if $\lim t_n = b$ and $\lim x_n = y$; this is very similar to the first case. That Q_y be in $L(\sigma)$ amounts to having that for all $t < b$, eventually $(t, y) \ll (t_n, x_n)$. First suppose $\{t_n\}$ approaches b and $\{x_n\}$ approaches y. Then eventually $(t_n, x_n) \in \bar{I}^+(t, y)$ (being an open neighborhood of (b, y)), so $(t, y) \ll (t_n, x_n)$, as required. Conversely, suppose for all $t < b$, eventually $(t, y) \ll (t_n, x_n)$. For all $t < b$, eventually $t_n > t$, so $\{t_n\}$ approaches b. Then for each j, an examination of M_j similar to above shows we must have, for any $t < b$, eventually $d_j(y_j, (x_n)_j) < \int_t^{t_n} f_j^{-\frac{1}{2}} < \int_t^b f_j^{-\frac{1}{2}}$. Then the same argument as above shows $\{x_n\}$ approaches y. $\square$

So how do our three examples of classical multiply warped product spacetimes fare in respect of Proposition 5.2?

First interior Schwarzschild (which is actually only conformal to a multiply warped product, but that's close enough): The spacelike factors are standard $\mathbb{R}^1$ and $\mathbb{S}^2$, complete. The warping factors are $f_1(r) = (2m/r - 1)^2$ and $f_2(r) = r^2(2m/r - 1)$. The future-endpoint of the interval in question, $(0, 2m)$, is at 0; thus, we need to examine $\int_0^c f_i(r)^{-\frac{1}{2}}\, dr$. For $i = 1$, this is $-c + 2m \ln(2m/(2m - c))$; for $i = 2$, this is $\pi/2 - \sin^{-1}(1 - c/m)$. Both are manifestly finite. Thus, the Schwarzschild singularity is spacelike with $\hat{}$-topology of $\mathbb{R}^1 \times \mathbb{S}^2$.

For Robertson-Walker spaces the only spacelike factor is K, a Riemannian manifold of constant curvature; usually this is taken to be $\mathbb{R}^3$, $\mathbb{S}^3$, or $\mathbb{H}^3$ (hyperbolic 3-space), but any quotient of these by a discrete group of isometries will do as well

and may serve just as appropriately for a cosmological model. Any such quotient will be complete. The warping factor is $r(t)^2$, where $r(t)$ scales the universe at time t. The integral condition to be satisfied is that $\int_c^\omega r(t)^{-1}\,dt < \infty$; this is precisely the condition that the spacetime be conformal to a finite (in the future) portion of the standard static spacetime $\mathbb{L}^1 \times K$ (i.e., respectively Minkowski space or the Einstein static space, in case K has 0 or positive constant curvature, assuming no identifications by isometries). For example, if the future-endpoint of time is ∞, then $r(t) = t$ fails, but $r(t) = t^{1+\epsilon}$ works for any $\epsilon > 0$; and if the future-endpoint of time is a finite ω, then $r(t) = \omega - t$ fails, but $r(t) = (\omega - t)^{1-\epsilon}$ works for any $\epsilon > 0$. When the integral condition is met, the boundary is spacelike and $\widehat{\ }$-homeomorphic to K.

For the Kasner spacetimes, the three spacelike factors are all standard $\mathbb{R}^1$, complete. The warping functions are $f_i(t) = t^{2p_i}$, with ∞ as the future-endpoint of the time interval. The integral condition is met for all cases except the "exceptional" ones where one of the numbers p_i is 1, and the other two are perforce 0; in that case, the spacetime is Rindler space, a portion of Minkowski space (with null boundary). Aside from the exceptional cases, the boundary is spacelike and $\widehat{\ }$-homeomorphic to $\mathbb{R}^3$.

5.3 Non-regular Boundaries from Embeddings.

Up to now, we have looked at examples of spacetimes with regular boundaries. But one wants to know how well the $\widehat{\ }$-topology measures up to non-regular boundaries, as well. One way that non-regular boundaries could conceivably come about (though perhaps not most naturally) is through embeddings: If $\phi : M \to N$ topologically embeds the spacetime M into the manifold N (of the same dimension), then M may acquire a boundary in the form of the boundary of $\phi[M]$ in N. In some circumstances this ϕ-boundary carries a natural extension of the chronology relation, defined for $x \in M$ and p in the ϕ-boundary by $x \ll p$ if there is a timelike curve in M with past endpoint at x and with ϕ-image approaching p in the future; and this might provide a future-completion for M. But there is no *a priori* reason to assume this boundary is regular.

For example: If M is a multiply warped product spacetime $(a, b) \times K$ (K the product of the spacelike factors), then one can easily embed M into the manifold $N = \mathbb{R} \times K$ via the obvious inclusion map; if b is finite, then this yields an obvious boundary for M of $\{b\} \times K$, agreeing with the Future Chronological Boundary in the case of a complete K and warping functions obeying the integral conditions—which is to say, in the case of a spacelike boundary. But consider some other embedding $\phi : M \to \mathbb{R}^1 \times K$; if, for instance, ϕ extends continuously to $\{b\} \times K$ but is not a homeomorphism there—say, $\phi(b, x) = \phi(b, y)$ for some $x \neq y$—then we obtain a new boundary for M, and we may wish to know how the $\widehat{\ }$-topology compares with the topology induced by ϕ.

The answer is that the two topologies are the same for a fairly wide class of "reasonable" embeddings ϕ; but the proof is not really dependent on the form of the spacetime, just on the fact that the it has only spacelike boundaries and ϕ extends continuously to the Future Chronological Boundary (in the $\widehat{\ }$-topology—which, for these spacetimes of Section 5.2, is $\{b\} \times K$). Accordingly, we will examine a general setting:

Let M be a strongly causal spacetime. Consider a topological embedding $\phi :$
$M \to N$, i.e., N is a manifold of the same dimension and ϕ is a homeomorphism

onto its image. We can think of the boundary in N of $\phi[M]$ as a sort of boundary for M: Identify M with its image under ϕ and just use the subspace topology for *closure*$(\phi[M])$. However, this could well include points that are not in any sense causally connected to the spacetime (example: $M = \{(x,t) \in \mathbb{L}^2 \mid |t| < x^2 \text{ and } 0 < x < 1\}$, $\phi : M \to \mathbb{L}^2$ is inclusion; the point $(0,0)$ is in the closure of M but not causally related to any point of M). To insure something that has the character of a future boundary, we will restrict ourselves to looking at boundary points with what amounts to a past in M: For any $p \in N$, let $I^-_M(p) = \{x \in M \mid \text{for some timelike curve } \gamma : [0,\infty) \to M, \gamma(0) = x \text{ and } \lim_{t\to\infty} \phi(\gamma(t)) = p\}$. Then the *future ϕ-boundary* of M, denoted $\partial^+_\phi(M)$, is defined as $\{p \in N - \phi[M] \mid I^-_M(p) \neq \emptyset\}$; and the *future ϕ-completion* of M, denoted $\bar{M}^+_\phi$, is defined as $M \cup \hat{\partial}^+_\phi(M)$, given the subspace topology from N by identifying M with its image under ϕ (this will sometimes be thought of as a subset of N, using $\phi[M]$ in place of M).

At the moment, $\bar{M}^+_\phi$ is just a topological space, albeit one that may have a future (topological) limit for various future-endless timelike curves in M. (If some timelike curves fail to have future endpoints in M^+_ϕ, then "future ϕ-completion" is something of a misnomer; but we will be considering only instances where this does not happen.) But we can also put a chronology relation on $\bar{M}^+_\phi$, extending the one on M: For x in M and p and q in $\partial^+_\phi(M)$, set $x \ll p$ if $x \in I^-_M(p)$; $p \ll x$ if for some $y \in M$ with $y \ll x$, $I^-_M(p) \subset I^-(y)$; and $p \ll q$ if for some $y \in I^-_M(q)$, $I^-_M(p) \subset I^-(y)$. Then it is not hard to check that the extended $\ll$ is a chronology relation on $\bar{M}^+_\phi$, with M chronologically dense in $\bar{M}^+_\phi$. (It therefore follows from Theorem 4.4 that in the $\hat{\ }$-topology on $\bar{M}^+_\phi$, as well as the subspace topology from N, M is topologically dense in $\bar{M}^+_\phi$, and that M, in its own topology, is homeomorphic to its image in $\bar{M}^+_\phi$, as a subspace of $\bar{M}^+_\phi$ with the $\hat{\ }$-topology.)

Note that if we assume M has only spacelike boundaries, then $p \ll a$ cannot occur for $p \in \partial^+_\phi(M)$ (and a anything in $\bar{M}^+_\phi$): If for some $y \in M$, $I^-_M(p)$ were in $I^-(y)$, then consider any $x \in I^-_M(p)$: There must be a timelike curve $\gamma : [0,\infty) \to M$ with $\gamma(0) = x$ and $\lim_{t\to\infty} \phi(\gamma(t)) = p$. Note that $P = I^-[\gamma]$ is an IP and that P is not any $I^-(z)$: For if it were, then z would be the future limit of γ (i.e., the future limit of any sequence of points on γ), hence its topological limit (Proposition 2.2); that would put $p = \phi(z)$, contradicting p being in $\partial^+_\phi(M)$. Therefore, $P \in \hat{\partial}(M)$ and must be inobservable. But $P \subset I^-_M(p)$, so if $I^-_M(p) \subset I^-(y)$, this contradicts the inobservability of P.

In fact, virtually the same argument now shows that all the elements of $\partial^+_\phi(M)$ are inobservable (if M has only spacelike boundaries): With $\bar{I}^-$ denoting the past in $\bar{M}^+_\phi$, we now know that for any $x \in M$, $\bar{I}^-(x) = I^-(x)$, and for any $p \in \partial^+_\phi(M)$, $\bar{I}^-(p) = I^-_M(p)$; also, we know that $\mathcal{IP}(\bar{M}^+_\phi)$ is the same as $\mathcal{IP}(M)$, as they have the same future chains. Now let p be any element of $\partial^+_\phi(M)$, and let P be any past component of p in $\bar{M}^+_\phi$. Then P contains some element x which has a timelike curve γ connecting it to p; in fact, $P = I^-[\gamma]$. As above, P is in $\hat{\partial}(M)$ and must be inobservable in $\hat{M}$. Therefore, P cannot be properly contained in any IP of M, hence, not in any IP of $\bar{M}^+_\phi$.

For M a spacetime with only spacelike boundaries, we will be considering that class of embeddings $\phi : M \to N$ which extend continuously to $\hat{\partial}(M)$, yielding

$\bar{\phi} : \hat{M} \to N$ (which will thus include embeddings of multiply warped product spacetimes continuously extending to $\{b\} \times K$). Note that in such a case, $\phi[M]$ is necessarily disjoint from $\bar{\phi}[\hat{\partial}(M)]$: For $P \in \hat{\partial}(M)$ generated by a timelike curve γ, P is the future limit of γ, hence, the topological limit of γ in $\hat{M}$; therefore, $\bar{\phi}(P) = \lim \phi(\gamma(t))$. If $\bar{\phi}(P) = \phi(x)$, then $\phi \circ \gamma$ eventually enters every neighborhood of $\phi(x)$, so the same is true in M: γ eventually enters every neighborhood of x, i.e., x is a future endpoint of γ in M. But that makes $P = I^-[\gamma]$ equal to $I^-(x)$, contradicting the assumption that P is in $\hat{\partial}(M)$. (This does not depend on M having spacelike boundaries.)

Note that we can readily identify the past components of any element p of $\partial^+_\phi(M)$: If P is in $\hat{\partial}(M)$ and $\bar{\phi}(P) = p$, then let γ be a generating timelike curve for P. As P is the future limit of γ, it is also its topological limit, so $\bar{\phi}(P)$ must be $\lim \phi(\gamma(t))$; therefore, $p = \lim \phi(\gamma(t))$, so all of γ lies in $I^-_M(p)$. Thus, $P \subset \bar{I}^-(p)$; as no IP can properly contain P, it must therefore be a past component of p. Furthermore, all past components of p arise in this way: The only IPs available are those of M, and only elements of $\hat{\partial}(M)$ can be maximal for lying in $\bar{I}^-(p)$. (Reason (see Figure 12): For any $I^-(x) \subset I^-_M(p)$, consider a future chain $\{x_n\}$ approaching x; for each n, there is a timelike curve c_n going, in essence, from x_n to p. These have a causal limit curve c beginning at x. Let $Q = I^-[c]$; this is an IP containing $I^-(x)$. For any $y \in Q$, there is some t with $y \ll c(t)$, so there is a sequence of numbers $\{t_n\}$ with $\{c_n(t_n)\}$ approaching $c(t)$. Since $c(t) \in I^+(y)$, eventually $c_n(t_n) \in I^+(y)$, i.e., $y \ll c_n(t_n)$. This puts y in $I^-_M(p)$. Thus, $Q \subset \bar{I}^-(p)$, so $I^-(x)$ is not a maximal IP in $\bar{I}^-(p)$.) If $P = I^-[\gamma]$ is one such past component, then for every t there is a timelike curve c_t from $\gamma(t)$ to (essentially) p. Let $P_t = I^-[c_t]$. For any $x \in P$, for t sufficiently large, $x \ll \gamma(t) = c_t(0)$, so $x \in P_t$, i.e., $x \ll P_t$; therefore P is the limit of $\{P_t\}$ in the $\hat{}$-topology (making use of the inobservability of P). It follows that $\bar{\phi}(P)$ is the limit of $\{\bar{\phi}(P_t)\}$. As shown above, for each t, $\bar{\phi}(P_t) = p$; this means $\bar{\phi}(P)$ is the limit of the constant sequence $\{p\}$, i.e., $\bar{\phi}(P) = p$.

Net result: For any $p \in \partial^+_\phi(M)$, the past components of p are precisely the elements of $\bar{\phi}^{-1}(p)$.

Not every such embedding produces a future completion in which the $\hat{}$-topology agrees with the ϕ-induced topology on $\bar{M}^+_\phi$. Consider lower Minkowski 2-space for M, i.e., $\{(x,t) \in \mathbb{L}^2 \mid t < 0\}$, and $\phi : M \to \mathbb{L}^2$ given by $\phi(x,t) = (t \tan^{-1} x, t)$. Then the only element of $\partial^+_\phi(M)$ is $(0,0)$ ($I^-_M((0,0))$ is all of M, but $I^-_M(t\pi/2, t) = \emptyset$ for $t < 0$). The past components of $(0,0)$ are all the IPs of the form $P_a = \{(x,t) \mid |x - a| < -t\}$ for $a \in \mathbb{R}$.

A sequence σ approaches the boundary point (0,0) in the $\hat{}$-topology if for every $\tau \subset \sigma$ there is a $\rho \subset \tau$ and some $a \in \mathbb{R}$ such that $P_a \in \mathcal{L}(\rho)$, i.e., $\{\rho(i)\}$ approaches $(a, 0)$ in the ordinary sense. For $\sigma(n) = (x_n, t_n)$, this is equivalent to $\{t_n\}$ approaching 0 and $\{x_n\}$ being bounded: If a set of numbers is bounded, then every subsequence has a subsubsequence with a limit; and if a set of numbers is unbounded, then it has a subsequence with no such subsubsequence. But in the ϕ-topology, $\{t_n\}$ going to 0 is all that's needed for convergence to (0,0); the two topologies differ on the convergence of $\{(n, -1/n)\}$.

It turns out that the crucial difference is whether or not the extension of ϕ to $\hat{\partial}(M)$ is proper onto its image. (A continuous map $f : X \to Y$ is defined to be proper if and only if for every compact $K \subset Y$, $f^{-1}(K)$ is compact. For second

$f : X \to Y$ is proper onto its image if and only if every sequence in X whose image under f converges to something in $f[X]$, has a convergent subsequence; thus, if f is injective and continuous, it is proper onto its image if and only if it is homeomorphic onto its image.)

If the entire extension $\bar{\phi} : \hat{M} \to N$ is proper onto its image (which can be identified with $\bar{M}^+_\phi$), then its restriction to the boundary, $\bar{\phi}_0 : \hat{\partial}(M) \to N$, is also proper onto its image (which is $\partial^+_\phi(M)$); this is a simple consequence of $\hat{\partial}(M)$ being closed in $\hat{M}$ (Proposition 2.6). But the converse fails: Consider M to be lower Minkowski 2-space as above, N the cylinder $\mathbb{R}^2/\!\sim$ for $(x,t) \sim (x, t+\pi/2)$, and $\phi : M \to N$ the map $\phi : (x,t) \mapsto [x, \tan^{-1} t]$ (the equivalence class): ϕ is a homeomorphism onto its image (which just barely fails to wrap once around the cylinder), and ϕ extends to a continuous $\bar{\phi} : \hat{M} \to N$ ($\bar{\phi}(P_x) = [x, 0]$ with notation as before) for which the restriction $\bar{\phi}_0 : \hat{\partial}(M) = \mathbb{R} \to N$ is proper onto its image; but the full $\bar{\phi} : \hat{M} \to N$ is not proper onto its image, as $\{\bar{\phi}(x, -n)\} = \{[x, -\tan^{-1} n]\}$ converges to $\bar{\phi}(P_x) = [x, 0]$, but $\{(x, -n)\}$ does not converge to P_x. And note that the ϕ-topology for $\bar{M}^+_\phi$ is that of the cylinder, while the $\hat{\ }$-topology is that of the closed half-plane (but $\partial^+_\phi(M)$ has the topology of $\mathbb{R}$ in either case).

Thus, we really have two separate theorems: One for the assumption that $\bar{\phi}$ is proper onto its image, informing the topology of $\bar{M}^+_\phi$; and one with the weaker assumption that $\bar{\phi}_0$ is proper onto its image, informing just the topology of $\partial^+_\phi(M)$. But we can handle them simultaneously.

Theorem 5.3. *Let M be a strongly causal spacetime with only spacelike boundaries, and let $\phi : M \to N$ be a topological embedding of M into a manifold of the same dimension, i.e., ϕ is homeomorphic onto its image.*

(1) *Suppose ϕ extends continuously (in the $\hat{\ }$-topology) to $\bar{\phi} : \hat{M} \to N$ so that $\bar{\phi}$ is proper onto its image; then the $\hat{\ }$-topology on $\bar{M}^+_\phi$ is the same as the ϕ-induced topology.*

(2) *Suppose that ϕ extends continuously to $\bar{\phi} : \hat{M} \to N$ so that the restriction $\bar{\phi}_0$ of $\bar{\phi}$ to $\hat{\partial}(M)$ is proper onto its image; then the $\hat{\ }$-topology on $\partial^+_\phi(M)$ is the same as the ϕ-induced topology.*

Proof. The method of proof is to show that $\bar{M}^+_\phi$ (or $\partial^+_\phi(M)$), in either topology, is homeomorphic to a quotient of $\hat{M}$ (or of $\hat{\partial}(M)$), as given in Theorem 4.8 (or in Corollary 4.9).

Note first that for any $P \in \hat{\partial}(M)$, $\bar{\phi}(P)$ is in $\bar{M}^+_\phi$: The IP P is generated by some timelike curve which can be parametrized as $\gamma : [0, \infty) \to M$, and P, being the future limit of γ, is also the topological limit of γ; thus, $\bar{\phi}(P) = \lim_{t\to\infty} \phi(\gamma(t))$, so $\gamma(0) \in I^-_M(\bar{\phi}(P))$. Furthermore, $\bar{\phi}(P) \in \partial^+_\phi(M)$, as was shown above.

Thus we have the continuous maps $\bar{\phi} : \hat{M} \to \bar{M}^+_\phi$ and $\bar{\phi}_0 : \hat{\partial}(M) \to \partial^+_\phi(M)$, with the $\hat{\ }$-topology on the domains and the topology induced by N (the ϕ-topology) on the targets. In fact, these maps are onto: Given a timelike curve $\gamma : [0, \infty) \to M$ with $\phi \circ \gamma$ converging to $p \in N$ and $p \notin \phi[M]$, let $P = I^-[\gamma]$. Then P must be in $\hat{\partial}(M)$, as otherwise $I^-[\gamma] = I^-(x)$ for some x, and that makes x the future endpoint of γ; then $\phi(x) = \lim_{t\to\infty} \phi(\gamma(t)) = p$, contradicting p being in $\partial^+_\phi(M)$. That allows us to consider $\bar{\phi}(P)$, which must be $\lim_{t\to\infty} \phi(\gamma(t)) = p$.

This gives us the continuous bijections $\widetilde{\phi} : \hat{M}/\sim \to \bar{M}^+_\phi$ and $\widetilde{\phi}_0 : \hat{\partial}(M)/\sim \to \partial^+_\phi(M)$, where $\sim$ is essentially the same equivalence relation for both maps: $P \sim Q$ for P and Q in $\hat{\partial}(M)$ and $\bar{\phi}(P) = \bar{\phi}(Q)$. Let $\pi : \hat{M} \to \hat{M}/\sim$ and $\pi_0 : \hat{\partial}(M) \to \hat{\partial}(M)/\sim$ denote the projections to equivalence classes. Then for any compact K in $\bar{M}^+_\phi$, $\widetilde{\phi}^{-1}[K] = \{\pi(z) \mid z \in \hat{M} \text{ and } \bar{\phi}(z) \in K\} = \pi[\,\bar{\phi}^{-1}[K]\,]$. We know $\bar{\phi}^{-1}[K]$ is compact if $\bar{\phi}$ is proper, whence its image under π is compact; thus, if $\bar{\phi}$ is proper, so is $\widetilde{\phi}$, and a proper, continuous bijection is a homeomorphism. Similarly, if $\bar{\phi}_0$ is proper, then $\widetilde{\phi}_0$ is a homeomorphism.

We now want to employ the theorems of Section 4. Note that $\bar{M}^+_\phi$ is generalized past-distinguishing: First, M is past-distinguishing. Second, if $x \in M$ and $p \in \partial^+_\phi(M)$ share a past component P, then P must be $I^-(x)$ and also P must be in $\bar{\phi}^{-1}(p)$; but that is impossible, as $\bar{\phi}^{-1}(p)$ consists of elements of $\hat{\partial}(M)$. And last, if p and q in $\partial^+_\phi(M)$ share a past component P then $\bar{\phi}(P) = p$ and $\bar{\phi}(P) = q$, so $p = q$. Also note that $\bar{M}^+_\phi$ is generalized future-complete: For any future chain c (which must lie wholly in M), let $P = I^-[c]$; if $P \in \hat{\partial}(M)$, then we must find a generalized future limit for c. Let $p = \bar{\phi}(P)$; then $P \in \bar{\phi}^{-1}(p)$, so P is a past component of p.

For case (1) we can now apply Theorem 4.8, which tells us that $\bar{M}^+_\phi$, in the $\hat{\ }$-topology, is a topological quotient of $\hat{M}$; more specifically, it says that $\bar{M}^+_\phi$ is $\hat{M}/\sim$ where $\sim$ identifies elements of $\hat{\partial}(M)$ if they have the same image under $\widetilde{\phi}^g$, where $\widetilde{\phi}^g : \hat{M} \to \bar{M}^+_\phi$ is the unique generalized future-continuous map such that $\widetilde{\phi}^g \circ \hat{\iota}_M = \phi$ (from Proposition 4.6(3)). This must be $\bar{\phi}$: Note that $\bar{\phi} \circ \hat{\iota}_M = \phi$; also, $\bar{\phi}$ is generalized future-continuous: For any future chain c, if its future limit is $x \in M$, then the same is true in $\bar{M}^+_\phi$ (same chronology relation); and if its future limit is $P = I^-[c] \in \hat{\partial}(M)$, then P is a past component of $p = \bar{\phi}(P)$, making p the generalized future limit of $\phi[c]$. Therefore, $\bar{\phi} = \widetilde{\phi}^g$, and the quotient of $\hat{M}$ given in Theorem 4.8 for the $\hat{\ }$-topology of $\bar{M}^+_\phi$ is precisely the same as the one above for the ϕ-topology.

For case (2) we apply Corollary 4.9. Note that for a spacetime, $\hat{\partial}^g(M) = \hat{\partial}(M)$, as a spacetime is regular. The maps and quotients are exactly the same as above, with the same result: The quotient of $\hat{\partial}(M)$ given in Corollary 4.9 for the $\hat{\ }$-topology of $\partial^+_\phi(M)$ is precisely the same as for the one above for the ϕ-topology. $\square$

It follows, for example, that not only is any generalized past-distinguishing and generalized future-completing boundary for interior Schwarzschild a quotient of $\mathbb{R}^1 \times \mathbb{S}^2$ in the $\hat{\ }$-topology for that completion (Corollary 4.9); but that, furthermore, any completion for interior Schwarzschild obtained by embedding it in another manifold so that the embedding continuously extends to the Future Chronological Boundary—the singularity—in a manner which maps the singularity properly onto its image, must be a quotient of $\mathbb{R}^1 \times \mathbb{S}^2$ in the induced topology from the embedding (Theorem 5.3). Thus is $\mathbb{R}^1 \times \mathbb{S}^2$ (the Future Chronological Boundary) universal for boundaries on interior Schwarzschild.

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